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# Kinetic theory analysis of generalized Rayleigh problem( Dissertation\_全文)

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# Kinetic Theory Analysis of Generalized Rayleigh Problem

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## Introduction

Many important problems in gasdynamics have successfully been studied by means of macroscopic equation of motion with the mass velocity, density and temperature as independent variables and involving various coefficient, e.g., viscosity, heat conduction etc. As the density of the gas is lowered, however, the mean free path (or mean collision period) of gas molecules becomes comparable to a distance (or time) in which we are interested from the physical standpoint. The macroscopic treatment then becomes invalid and we must resort to a more fundamental and general microscopic formalism. The Boltzmann equation is generally accepted to be a fundamental equation of rarefied gasdynamics for the entire range of conditions from continuum to free molecular flow. However, the solution of this equation is, in general, very difficult even in cases corresponding to physically simplest situations.

One of the fundamental problems to be investigated in rarefied gasdynamics will be the response of the gas when the condition (e.g. velocity, temperature, etc.) of a solid boundary in it changes appreciably in a short time (comparable to mean collision period of gas molecules). As a simplest case of such phenomena, we consider in the present paper the generalized Rayleigh problem. Namely, we investigate the behavior of a dilute gas bounded by an infinite flat wall when the wall is impulsively set into uniform

motion parallel to itself and at the same time its temperature is changed suddenly.

We use throughout the present paper the B-G-K model of Boltzmann equation suggested by Krook et al.<sup>1</sup>, in which collision integral is replaced by mathematically simpler term retaining essential features of collision. This equation has hitherto been applied successfully to several flow problems, e.g. propagation of plane wave by Krook et al.<sup>1</sup>, linearized Couette flow by Willis<sup>2</sup> and structure of shock wave by Liepmann et al.<sup>3</sup>. Short account of this model in relation to the standard Boltzmann equation is given in Appendix 2.

We further assume that the velocity and temperature jump of the wall are so small that the fundamental equation and the boundary conditions may be linearized. Then it turns out that the effect of wall motion is never coupled with that of the variation of wall temperature. Therefore, the two effects may be analyzed separately and superimposed to form the answer to the generalized Rayleigh problem. Part 1 of the present paper concerns mainly with the effect of the motion of the wall and in Part 2 is studied the problem of wall temperature variation.

M. Shibata and the author<sup>4</sup> have quite recently investigated how the initial plane of discontinuity in tangential flow velocity diffuses basing on the B-G-K model. This has

obvious resemblance to the Rayleigh problem, but an essential difference of no presence of solid boundary in the gas. Comparison of the two results may serve better understanding of the dynamics of dilute gas and a brief summary of the study is given in Appendix 3.

The author expresses his cordial thanks to Professor Kō Tamada for his kind guidance throughout this work.

Part 1

Kinetic Theory Analysis of Linearized Rayleigh  
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## Kinetic Theory Analysis of Linearized Rayleigh Problem

### Summary

The problem of an infinite flat plate set impulsively into uniform motion in its own plane in an infinite mass of fluid is discussed using the Bhatnager, Gross and Krook model of the Boltzmann equation. The velocity of the plate is assumed to be small and the equations as well as the boundary conditions are linearized. The velocity field and the stress on the plate are obtained for both short and long times. For short times the solution represents a perturbation to the linearized free molecular flow. It involves also at long times essential difference from the classical slip flow near the boundary. Numerical value of slip coefficient is calculated.

## § 1. Introduction

The investigation of the dynamics of a gas bounded by an infinite plane which is set impulsively into uniform motion along its plane was first treated by Rayleigh.<sup>6</sup> He considered that the fluid is viscous and incompressible and that its motion is describable by the Navier-Stokes equation subject to the non-slip boundary condition. Removal of the restriction that there is no slip in flow velocity at the boundary was first made by Schaaf.<sup>7</sup> For a compressible fluid the classical Rayleigh problem has been examined by Howarth,<sup>8</sup> Van Dyke,<sup>9</sup> Stewartson,<sup>10</sup> and Hanin.<sup>11</sup> It is clear, however, that the Navier-Stokes equations are inadequate for the description of the gas shortly after the plane has been set into motion. Howarth in fact questions the significance of his solution, based on the Navier-Stokes equations, for the period of time which is small compared with the time between molecular collisions. Yang and Lees<sup>12</sup> investigated the linearized\* Rayleigh problem by means of the Grad's thirteen moment approximation.<sup>13</sup> This method was however found to be inadequate at short times, predicting incorrect values for both the initial stress and flow velocity. Gross and Jackson<sup>14</sup> also investigated the problem using the half-range method of solution of the

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\* "Linearized" means that the velocity is assumed to be so small that the equations as well as the boundary conditions may be linearized.

Boltzmann equation. This method gives exact results for the initial flow velocity and stress at the boundary, but seems to be still inadequate to describe the propagation of the disturbance.

We here try to obtain a complete solution of this problem with B-G-K statistical model,<sup>1</sup> which should prove to be very instructive, both for an understanding of the short time behavior of Rayleigh flow and as a means of evaluation of the B-G-K model itself and other kinetic theory and continuum representations. It turns out that no explicit form of the velocity distribution function is needed to obtain mean quantities such as flow velocity and stresses. An integral equation for mean velocity of gas is derived (Eq. (21)). Stresses are then calculated from the velocity field (Eq. (25)). The velocity and stress on the plate are obtained for both short and long times. For short times the solution represents a perturbation to the linearized free molecular flow ((36), (40) and (42)). It involves also at long times essential difference from the classical slip flow in a thin layer adjacent to the boundary, with thickness of the order of mean free path ((57) and (59)). The flow field outside this layer may be described by the slip flow solution with appropriate values of kinematic viscosity and slip coefficient ((44) and (55)). Some of the results obtained here are compared with the corresponding results from other methods such as the free molecular

flow approach and the method of half-range moment due to Gross et al.<sup>14</sup> (Figs. 2, 3 and 4). There is found considerable difference between the results by Gross et al., and ours for the initial rate of reduction due to molecular collision, of both the slip velocity and the stress on the plate (Figs. 3 and 4). Our result for the stress on the plate for long times agrees with the classical results of Rayleigh or Schaaf up to the order  $t^{-\frac{1}{2}}$  where  $t$  is time and it gives about 10% larger value than the result obtained by Gross et al. The latter involves a deviation from the classical result due to a molecular boundary effect even in the order  $t^{-\frac{1}{2}}$  (Fig. 4).



## § 2. Fundamental Equation

We consider the motion of the fluid bounded by an infinite flat plate which is initially in equilibrium with the bounding fluid and set at some instant impulsively into uniform motion in its own plane. We choose a Cartesian coordinate system  $(x, y, z)$  fixed in the space in such a way that the  $x$ -axis is in the direction of motion of the plate and the  $y$ -axis normal to the plate. The fluid is in the region  $y > 0$  (Fig. 1). It may then be assumed that the field quantities are independent of  $x$  and  $z$  ( $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} = 0$ ).

We use here the Boltzmann equation with the collisional model suggested by Krook et al. Let  $m$  be the mass of a molecule,  $\mathbf{v} = (u, v, w)$  the velocity of a molecule and  $f$  the mass density of molecules in 6-dimensional space  $(x, y, z; u, v, w)$ . Then, the Boltzmann equation for the present problem may be written in a general form

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial y} = m G(f) - f A(f) \quad (1)$$

The right-hand side is the well-known collision integral, in which  $m G(f)$  and  $f A(f)$  are respectively proportional to the numbers of molecules gained and lost per unit volume and unit time at  $(x, y, z, t, u, v, w)$ . The main difficulty in handling the full Boltzmann equation arises from

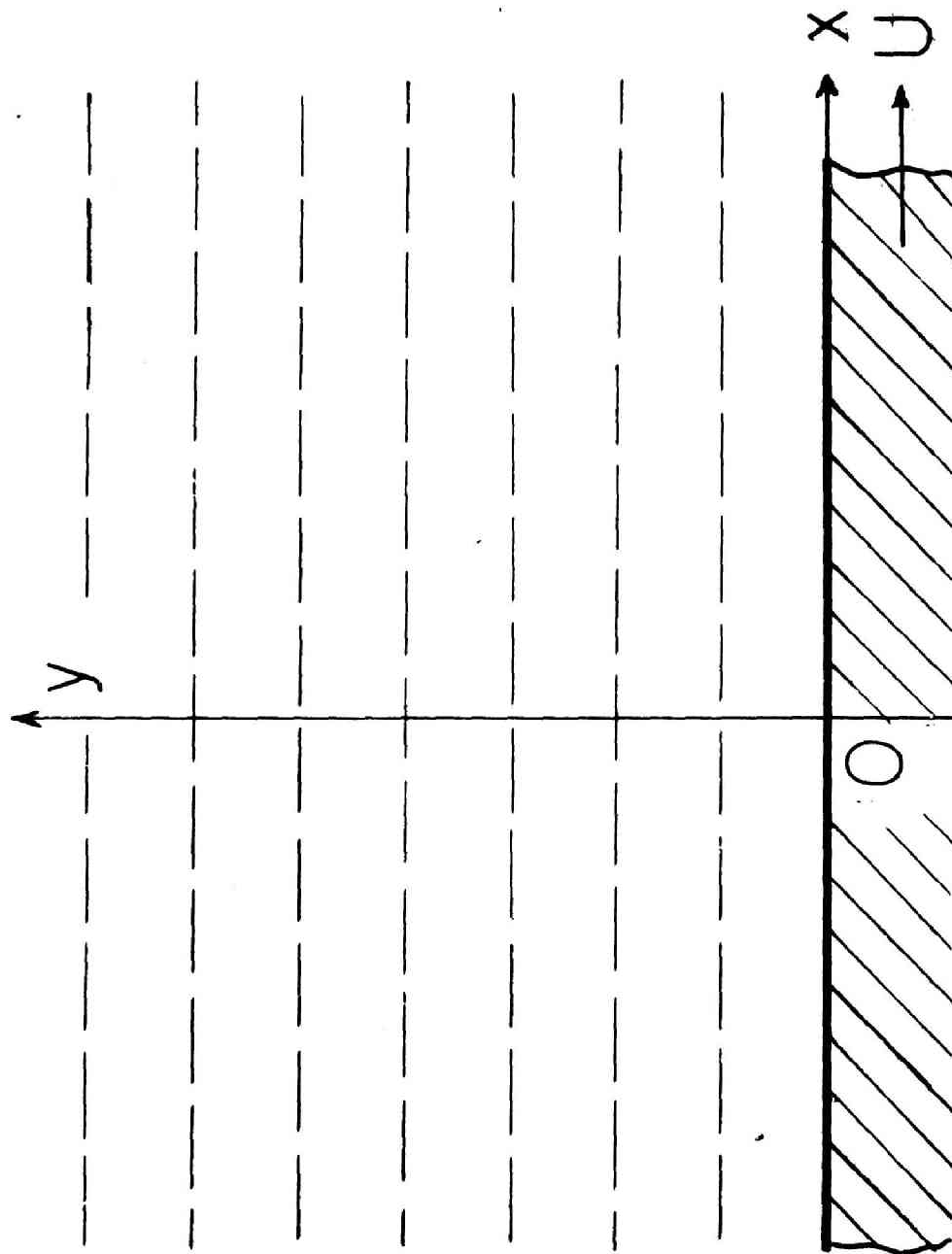


Fig. 1. Coordinate system

the complicated nature of the collision term. Thus, Krook et al. suggested so-called B-G-K statistical model which is much simpler mathematically but conforms to the conservation laws of mass, momentum and energy, and at the same time represents certain essential features of collisions. The collision term in B-G-K model is simplified as follows:

$$\left. \begin{aligned} \mathcal{A}(f) &= \frac{\rho}{\chi} \\ m \mathcal{G}(f) &= \frac{\rho^2}{\chi} \Phi \\ \Phi &= \left[ \frac{m}{2\pi kT} \right]^{3/2} \exp \left\{ -\frac{m}{2kT} (v - \mathbf{g})^2 \right\} \end{aligned} \right\} \quad (2)$$

where

$$\rho = \int f \, dv \quad (3)^*$$

$$\mathbf{g} = \frac{1}{\rho} \int \mathbf{v} f \, dv \quad (4)$$

$$\frac{3kT}{m} = \frac{1}{\rho} \int (v - \mathbf{g})^2 f \, dv \quad (5)$$

and  $\rho$ ,  $\mathbf{g} = (g_x, g_y, 0)$ ,  $T$ , and  $k$  are, respectively, the local density, gas velocity, temperature, and Boltzmann constant. The number  $\chi$  is a parameter which is related to collision time and in general may

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\* Integration  $\int (...) \, dv$  is hereafter carried over the whole molecular velocity space unless otherwise stated.

depend on the state of gas. In the present study, we assume  $\chi$  to be constant for simplicity. It is easy to see that the collision term simplified as (2) possesses the same five collisional invariants  $m$ ,  $mv$ ,  $mv^2$ , as the exact one. The connection of the B-G-K kinetic equation with the standard Boltzmann equation is discussed by Kogan and Liepmann et al.<sup>15,3</sup> (Appendix 2).

To describe the interaction of the gas with the boundary, we shall assume the diffuse reflection. Namely, i) the reflected molecules have a Maxwellian distribution characterized by the velocity and temperature of the plate, and ii) the net mass flow at the plate vanishes ( $\int v f dv = 0$  at  $y = 0$ ). Also, the temperature of the plate is kept constant at  $T_0$ .

In the present study, the velocity of the plate  $U$  is assumed to be much less than the velocity of sound in the gas so that the fundamental equations as well as the boundary conditions may be linearized. To carry out the linearization, we write

$$f = f_0 F(1 + \phi) \quad (6)$$

$$\rho = \rho_0 (1 + \omega) \quad (7)$$

$$T = T_0 (1 + \tau) \quad (8)$$

$$F = \left(\frac{h}{\pi}\right)^{\frac{3}{2}} \exp\{-h(u^2 + v^2 + w^2)\} \quad \left. \vphantom{\left(\frac{h}{\pi}\right)^{\frac{3}{2}}} \right\} \quad (9)$$

$$h = \frac{m}{2kT_0}$$

where  $T_0$  is the initial temperature of the system (or the wall temperature at any instant) and  $\rho_0 F$  is the Maxwellian or equilibrium initial distribution with density

$\rho_0$ , temperature  $T_0$  and zero macroscopic velocity.

Since we are concerned with the case of small plate velocity,  $\phi$  (and so  $\omega$ ,  $q$  and  $\tau$ ) may be considered to be of the first order of smallness. If we neglect all the higher order terms, Eqs (1) to (5) become

$$\frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial y} = \lambda \left\{ \omega - \phi + 2kvq + \tau \left( kv^2 - \frac{3}{2} \right) \right\} \quad (10)$$

$$\omega = \int \phi F dv \quad (11)$$

$$q = \int v \phi F dv \quad (12)$$

$$\frac{3}{2k}(\omega + \tau) = \int v^2 \phi F dv \quad (13)$$

where  $\lambda \equiv (\rho_0/\chi)$  is a constant (collision frequency) related to the mean free path  $l$  as  $l = (2/\lambda \sqrt{\pi k})$ .

In the present case of constant wall temperature,  $\omega$ ,  $q$  and  $\tau$  may be set all equal to zero. This will be verified later. We then have

$$\frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial y} = \lambda (-\phi + 2h u g_x) \quad (10a)$$

$$g_x = \int u \phi F dv \quad (12a)$$

The boundary condition at the wall is that the reflected molecules should have a Maxwellian distribution with mean velocity  $U$  and temperature  $T_0$ . That is,

$$f = \rho_0 (1 + \phi_i(t)) \exp[-h\{(u-U)^2 + v^2 + w^2\}]$$

where  $\phi_i(t)$ , a function of  $t$  alone, is related to  $f(v < 0)$  (and thus to  $\phi(v < 0)$ ) by the condition of no absorption at the wall ( $\int v f dv = 0$  at  $y=0$ ). Since  $U$  is very small, the above condition may also be linearized, obtaining:

$$\left. \begin{aligned} \phi &= \phi_i(t) + 2h u U \\ \phi_i(t) &= -2\sqrt{\pi}h \int_{v < 0} v \phi_{y=0} F dv \end{aligned} \right\} \quad (14)$$

It will be shown later [cf.(19)] that  $\phi_i(t)$  may be taken zero in the present case. Thus, the boundary condition at the wall becomes

$$\phi = 2h u U \quad (v > 0) \quad \text{at } y=0 \quad (14a)$$

Further, the condition at infinity is seen to be

$$\phi \rightarrow 0 \quad (v < 0) \quad \text{as } y \rightarrow \infty \quad (14b)$$

while the initial condition of the problem is

$$\phi = 0 \quad \text{at } t = 0 \quad (15)$$

To solve the system of Eqs.(10a) and (12a), we apply the Laplace transformation defined in the form

$$\hat{M}(s, y) = \int_0^{\infty} M(t, y) \exp(-st) dt \quad (16)$$

where  $s$  is the transform variable and  $\hat{M}(s, y)$  the transform of a function  $M(t, y)$  defined for all positive values of  $t$ . The Laplace transforms of (10a) and (12a) are written in the forms

$$\frac{\partial \hat{\phi}}{\partial y} + \frac{s+\lambda}{v} \hat{\phi} = \frac{2k\lambda u}{v} \hat{\phi}_x \quad (17)$$

$$\hat{\phi}_x = \int u \hat{\phi} F dv \quad (18)$$

Solving (17) for  $\hat{\phi}$  under the conditions (14a, b), we obtain

$$\hat{\phi} = \frac{2k\lambda u}{v} \int_{\infty}^y \hat{\phi}_x \exp\left\{\frac{s+\lambda}{v}(y_0 - y)\right\} dy_0 \quad (v < 0) \quad (19)$$

$$\hat{\phi} = \frac{2\kappa u U}{\lambda} \exp\left\{-\frac{\lambda+\lambda}{v} y\right\} + \frac{2\kappa\lambda u}{v} \int_0^y \hat{g}_x \exp\left\{-\frac{\lambda+\lambda}{v}(y-y_0)\right\} dy_0 \quad (v > 0) \quad (20)$$

It will be seen that  $\hat{\phi}$  and hence  $\phi$  are odd functions of  $u$ . Eliminating  $\hat{\phi}$  from (18), (19) and (20), we find that the flow velocity  $\hat{g}_x$  satisfies the following integral equation:

$$\frac{\hat{g}_x}{U} = \frac{1}{\sqrt{\pi} \lambda} J_0(\sqrt{\kappa}(\lambda+\lambda)y) + \lambda\sqrt{\frac{\kappa}{\pi}} \int_0^\infty \left(\frac{\hat{g}_x}{U}\right) J_{-1}(\sqrt{\kappa}(\lambda+\lambda)|y-y_0|) dy_0 \quad (21)$$

where the functions  $J_n'$  are defined by the integral

$$J_n(\xi) = \int_0^\infty \xi^n \exp\left\{-\left(\xi^2 + \frac{\xi}{\xi}\right)\right\} d\xi \quad (22)$$

and have been investigated by Faxén<sup>16</sup> and Abramowitz<sup>17</sup> (Appendix 1).

The shear stress  $p_{xy}$  is written, by its definition, as:

$$\begin{aligned} \frac{p_{xy}}{\rho_0 U^2} &= \frac{1}{U^2} \int (u - g_x) v F(1 + \phi) dv \\ &= -\frac{1}{U^2} \int u v \hat{\phi} F dv \end{aligned} \quad (23)$$



The corresponding Laplace transform is

$$\frac{\hat{p}_{xy}}{\rho_0 U^2} = \frac{1}{U^2} \int u v \hat{\phi} F dv \quad (24)$$

Substituting (19) and (20) into (24), we obtain the formula giving shear stress  $\hat{p}_{xy}$  in terms of the gas velocity  $\hat{g}_x$ . Namely,

$$\begin{aligned} \frac{\hat{p}_{xy}}{\rho_0 U^2} = & \frac{1}{\sqrt{\pi k} U s} J_1(\sqrt{k}(s+\lambda)y) \\ & + \frac{\lambda}{\sqrt{\pi} U} \int_0^y \left( \frac{\hat{g}_x}{U} \right) J_0(\sqrt{k}(s+\lambda)(y-y_0)) dy_0 \\ & + \frac{\lambda}{\sqrt{\pi} U} \int_{-\infty}^y \left( \frac{\hat{g}_x}{U} \right) J_0(\sqrt{k}(s+\lambda)(y_0-y)) dy_0 \end{aligned} \quad (25)$$

Especially, we have at  $y=0$  the result

$$\left( \frac{\hat{p}_{xy}}{\rho_0 U^2} \right)_{y=0} = \frac{1}{2\sqrt{\pi k} U s} - \frac{\lambda}{\sqrt{\pi} U} \int_0^{\infty} \left( \frac{\hat{g}_x}{U} \right) J_0(\sqrt{k}(s+\lambda)y_0) dy_0 \quad (26)$$

Considering that  $\hat{\phi}$  (or  $\phi$ ) is an odd function of  $u$ , the normal stresses  $p_{xx}$  etc. and the pressure  $p$  are easily seen to be constant, i.e.

$$p_{xx} = p_{yy} = p_{zz} = p = \frac{\rho_0}{2k} \quad (27)$$

We now introduce the following non-dimensional quantities

$$\left. \begin{aligned} \eta &= \sqrt{h} \lambda y, \quad (\eta_0 = \sqrt{h} \lambda y_0), \quad \sigma = \frac{A}{\lambda} \\ Q &= \lambda \left( \frac{\hat{\rho}_x}{U} \right), \quad \text{and} \quad P_{xy} = \lambda \left( \frac{\hat{p}_{xy}}{\rho_0 U^2} \right) \end{aligned} \right\} (28)$$

With these variables Eqs. (21), (25) and (26) are rewritten in the forms:

$$Q = \frac{1}{\sqrt{\pi}} \left\{ \frac{J_0((\sigma+1)\eta)}{\sigma} + \int_0^\infty Q J_1((\sigma+1)|\eta-\eta_0|) d\eta_0 \right\} \quad (29)$$

$$P_{xy} = \frac{1}{\sqrt{\pi h} U} \left[ \frac{J_1((\sigma+1)\eta)}{\sigma} + \int_0^\eta Q J_0((\sigma+1)(\eta-\eta_0)) d\eta_0 \right. \\ \left. + \int_\infty^\eta Q J_0((\sigma+1)(\eta_0-\eta)) d\eta_0 \right] \quad (30)$$

$$(P_{xy})_{\eta=0} = \frac{1}{\sqrt{\pi h} U} \left[ \frac{1}{2\sigma} - \int_0^\infty Q J_0((\sigma+1)\eta_0) d\eta_0 \right] \quad (31)$$

Before proceeding to the solution of Eq.(29), a few remarks will be made about the fields of  $\omega$ ,  $\rho_y$  and

$\tau$ . In deriving Eq.(21) and the boundary condition (14a), we have assumed that  $\omega$ ,  $\rho_y$ ,  $\tau$  and  $\phi_i(t)$  are all equal to zero. This is easily verified from the fact that

$\hat{\phi}$  (or  $\phi$ ) is an odd function of  $u$  (Eqs.(19) and (20)) and the definitions of  $\omega$ ,  $g_y$ ,  $\tau$  and  $\phi_i(t)$  (Eqs.(11), (12), (13) and (14)). Secondly, the wall temperature is assumed, in the present study, to be kept at a constant value  $T_0$ . It can be seen however that even in a more general problem in which the plate velocity  $U$  and its temperature  $T_w$  are arbitrary given functions of  $t$ , the field  $g_x$  and the fields  $\omega$ ,  $g_y$  and  $\tau$  are not coupled so long as the conditions for linearization ( $\sqrt{Re} U \ll 1$ ,  $|T_0 - T_w| \ll T_0$  etc.) are fulfilled (§ 4.). That is, we obtain two sets of equations which are not coupled with each other, one concerning  $g_x$  and  $U$  only and the other containing  $\omega$ ,  $g_y$ ,  $\tau$  and  $T_w$  alone. Our solution given below therefore may represent the velocity field of more general problem in which the wall temperature  $T_w$  is not necessarily equal to  $T_0$  for  $t > 0$ . The problem in which the wall temperature jumps suddenly from  $T_0$  to  $T_w$  (slightly different from  $T_0$ ) seems also to be of considerable interest and will be treated in Part 2.

### § 3. Velocity Field and Stress

In this section we shall first study the motion of the gas by solving the integral equation (21) or (29) and then calculate the stress on the plate by the formula (31).

#### 3.1 Solution Suitable for Small Values of the Time

For short times (  $\lambda t \ll 1$  or  $\sigma \gg 1$  ), the motion of the gas is expected to be close to the free molecular flow (  $\lambda = 0$  in (21)). This can be seen by observing that the function  $J_1$  in Eq.(29) has the property

$$J_1((\sigma+1)|\eta-\eta_0|) \longrightarrow \frac{\sqrt{\pi}}{\sigma+1} \delta(\eta-\eta_0) \quad (\sigma \rightarrow \infty)$$

where  $\delta(\eta-\eta_0)$  is Dirac's delta function. Thus, a first approximation for  $Q$  may be obtained by substituting the first term which represents the free molecular flow for  $\sigma \rightarrow \infty$  into  $Q$  of the second term on the right-hand side of Eq.(29). That is,

$$Q = \frac{1}{\sqrt{\pi}} \sigma \left[ J_0(\alpha \eta) + \frac{1}{\sqrt{\pi}} I \right] \quad \left. \begin{array}{l} \text{with} \\ I = \int_0^\infty J_0(\alpha \eta_0) J_1(\alpha |\eta - \eta_0|) d\eta_0 \quad \alpha = \sigma + 1 \end{array} \right\} (32)$$

The task is thus to evaluate the integral in (32).

a) Near field to the plate ( $\sqrt{k} y/t \ll 1$  or  $\sigma\eta \ll 1$  )

The integral may be split into two parts,

$$I = \left( \int_0^\eta + \int_\eta^\infty \right) J_0(\alpha\eta_0) J_1(\alpha|\eta-\eta_0|) d\eta_0$$

In the first integral, the quantities  $\alpha\eta_0$  and  $\alpha(\eta-\eta_0)$  are both so small that we may use for  $J_0(\alpha\eta_0)$  and  $J_1(\alpha(\eta-\eta_0))$  respective series expansions\* around  $\eta_0 = 0$  and  $\eta_0 = \eta$ , and integrate term by term. Thus we have

$$\int_0^\eta J_0(\alpha\eta_0) J_1(\alpha(\eta-\eta_0)) d\eta_0 \cong \frac{\sqrt{\pi}}{2} \eta \left[ -\log \alpha\eta + \left( 1 - \frac{3}{2}\gamma \right) \right] \quad (33)$$

where  $\gamma = 0.577\dots$  is Euler's constant. To evaluate the second integral, we insert for  $J_n$  its definition, change the order of integration and carry out some manipulation, obtaining

$$\begin{aligned} & \int_\eta^\infty J_0(\alpha\eta_0) J_1(\alpha(\eta_0-\eta)) d\eta_0 \\ &= \frac{1}{\alpha} \iint_0^\infty \frac{\xi}{\xi+\zeta} \exp \left[ -(\zeta^2 + \xi^2) - \frac{\alpha\eta}{\xi} \right] d\zeta d\xi \\ &\cong \frac{\pi}{8} \frac{1}{\alpha} - \eta \sqrt{\frac{\pi}{2}} \log(\sqrt{2}+1) \end{aligned} \quad (34)$$

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\* Appendix 1

Combining (33) with (34), we have the result

$$I \cong \frac{\pi}{8} \frac{1}{\alpha} + \gamma \frac{\sqrt{\pi}}{2} \left[ -\log \alpha \gamma + \left\{ 1 - \frac{3}{2} \gamma - \sqrt{2} \log(\sqrt{2} + 1) \right\} \right] \quad (35)$$

Taking the Laplace inversion of (32) with the aid of (28) and (35), we find the flow velocity near the plate as follows:

$$\begin{aligned} \frac{g_x}{U} \cong & \frac{1}{2} - \sqrt{\frac{R}{\pi}} \left( \frac{y}{t} \right) \\ & + \lambda t \left[ \frac{1}{8} + \frac{1}{2} \sqrt{\frac{R}{\pi}} \left( \frac{y}{t} \right) \left\{ \log \frac{\sqrt{R} y}{t} + 1 + \frac{\gamma}{2} - \sqrt{2} \log(\sqrt{2} + 1) \right\} \right] \\ & \frac{\sqrt{R} y}{t} \ll 1 \quad (36) \end{aligned}$$

In particular, we have on the plate

$$\left( \frac{g_x}{U} \right)_{y=0} \cong \frac{1}{2} + \frac{\lambda t}{8} \quad (37)$$

It will be seen that the flow is more accelerated than the free molecular flow ( $\lambda = 0$ ) on and near the plate.

This is because incoming molecules have obtained the average velocity in x-direction due to collision with outgoing molecules (Fig.2 and 3).

b) Far field from the plate ( $\sqrt{R} y/t \gg 1$  or  $\alpha \gamma \gg 1$ )

Breaking the integral in (32) into three parts:

$$I = \left( \int_0^{\frac{1}{2}\gamma} + \int_{\frac{1}{2}\gamma}^{\gamma} + \int_{\gamma}^{\infty} \right) J_0(\alpha \eta) J_1(\alpha |\eta - \eta_0|) d\eta$$

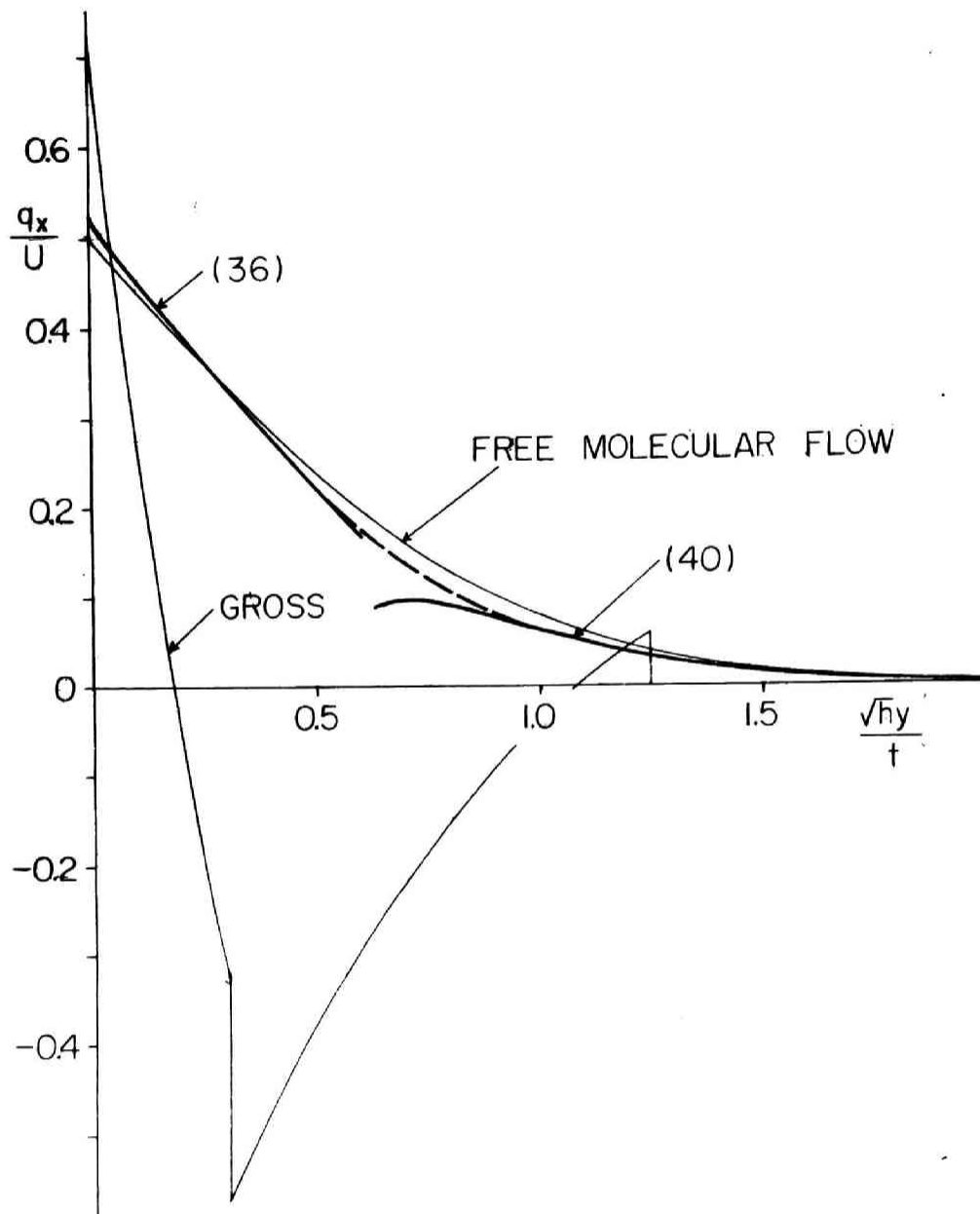


Fig. 2. The distribution of the mean velocity in the gas  
 $(\lambda t = 0.2)$

GROSS: Method of half-range moment by Gross and Jackson. <sup>14</sup>

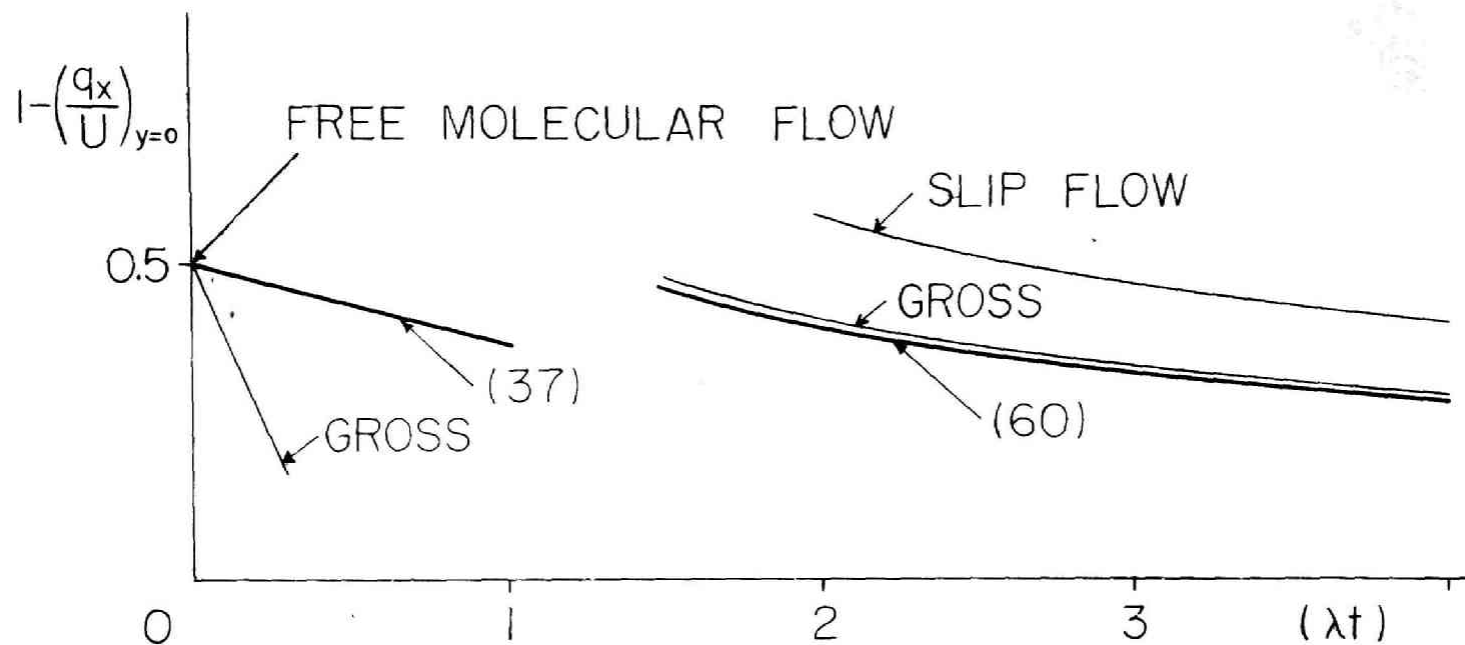


Fig. 3. The slip velocity on the plate as a function of time.

GROSS: Method of half-range moment by Gross and Jackson.<sup>14</sup>

SLIP FLOW:  $\nu = 1/2\hbar\lambda$ ,  $\delta = 1.801/\sqrt{\pi\hbar\lambda}$  [(59)].



and integrating by parts, we have

$$\begin{aligned}
 I = & \frac{2J_0(0)}{\alpha} J_0(\alpha\eta) - \frac{1}{\alpha} \left\{ J_0\left(\frac{\alpha\eta}{2}\right) \right\}^2 \\
 & + 2 \int_0^{\frac{\eta}{2}} J_0(\alpha\eta_0) J_{-1}(\alpha(\eta-\eta_0)) d\eta_0 \\
 & - \int_{\frac{\eta}{2}}^{\infty} J_{-1}(\alpha\eta_0) J_0(\alpha(\eta_0-\eta)) d\eta_0
 \end{aligned}$$

Integrating by parts further and taking into account the asymptotic behavior of  $J_n$  for  $\alpha\eta \gg 1$ \*, we get

$$I \cong \frac{1}{\alpha} \left[ 2 J_0(0) J_0(\alpha\eta) + J_1(0) J_{-1}(\alpha\eta) + \dots \right] \quad (38)$$

Thus for  $\alpha\eta \gg 1$ , Eq.(32) is reduced to

$$Q \cong \frac{1}{\sqrt{\pi}} \left[ \frac{1}{\sigma+1} J_0((\sigma+1)\eta) + \frac{2}{(\sigma+1)^2} \left\{ J_0((\sigma+1)\eta) + \dots \right\} \right] \quad (39)$$

from which we obtain the macroscopic velocity of the gas in the form:

$$\begin{aligned}
 \frac{\partial x}{U} \cong & \frac{t}{2\sqrt{\pi\kappa}y} \left[ 1 - \frac{1}{2} \left( \frac{t}{\sqrt{\kappa}y} \right)^2 - \lambda t \left\{ 1 - \frac{3}{2} \left( \frac{t}{\sqrt{\kappa}y} \right)^2 \right\} \right] \\
 & \times \exp \left[ - \left( \frac{\sqrt{\kappa}y}{t} \right)^2 \right] \quad \frac{\sqrt{\kappa}y}{t} \gg 1 \quad (40)
 \end{aligned}$$

---

\* Appendix 1

For large  $y$ , the flow is less accelerated than the free molecular flow ( $\lambda = 0$ ), because reflected molecules lose some of their average velocity in x-direction by collision with molecules going towards the wall before they reach the point in question (Fig.2).

We next calculate the stress on the plate. We insert, as a first approximation, the first term on the right-hand side of (29) into Q in (31). Then,

$$\begin{aligned}
 (P_{xy})_{y=0} &\cong \frac{1}{\sqrt{\pi} h U \sigma} \left[ \frac{1}{2} - \frac{1}{\sqrt{\pi}} \int_0^{\infty} J_0^2(\alpha \eta_0) d\eta_0 \right] \\
 &\cong \frac{1}{2\sqrt{\pi} h U} \\
 &\quad \times \left[ \frac{1}{\alpha} - \frac{1}{2} \left\{ 1 - \frac{1}{\sqrt{2}} \log(\sqrt{2}+1) \right\} \frac{1}{\sigma(\sigma+1)} \right] \quad (41)
 \end{aligned}$$

Inverting (41), we obtain the required stress in the form

$$\left( \frac{P_{xy}}{\rho U^2} \right)_{y=0} \cong \frac{1}{2\sqrt{\pi} h U} \left[ 1 - \frac{\lambda t}{2} \left\{ 1 - \frac{1}{\sqrt{2}} \log(\sqrt{2}+1) \right\} \right] \quad (42)$$

$$1 - \frac{1}{\sqrt{2}} \log(\sqrt{2}+1) \doteq 0.377$$

$$\lambda t \ll 1$$

where the second term represents an effect of molecular collision resulting in the reduction of the stress on the plate (Fig.4).

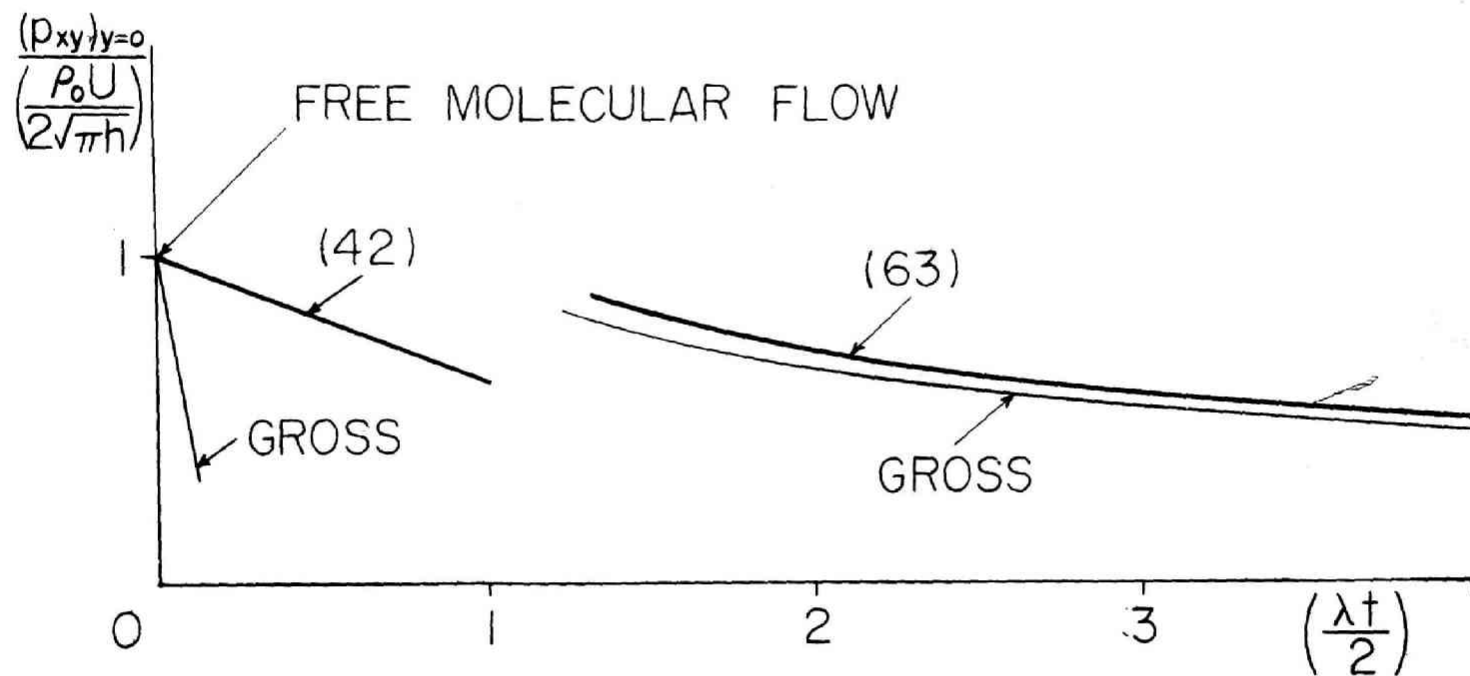


Fig. 4. The shear stress on the plate as a function of time.

GROSS: Method of half-range moment by Gross and Jackson.<sup>14</sup>

### 3.2 Solution for Large Values of the Time

Before solving the equation (29) for  $\lambda t \gg 1$ , we shall first give formal derivation from (21) under  $\lambda \gg 1$  of the Navier-Stokes equation and slip boundary condition. By this consideration we may clarify some defects of the classical theory.

Carrying out integration by parts successively on the right-hand side of (21) and neglecting the term involving  $J_n(\sqrt{\kappa}(\lambda + \lambda)y)$  on account of its asymptotic behavior\* for  $\sqrt{\kappa}(\lambda + \lambda)y \gg 1$  (which does not hold for  $y \ll 1$ ), we have

$$\lambda \hat{\theta}_x \cong \frac{1}{2\kappa\lambda(1+\lambda/\lambda)^2} \left[ \frac{\partial^2 \hat{\theta}_x}{\partial y^2} + \frac{2}{\sqrt{\pi\kappa}\lambda(1+\lambda/\lambda)} \right. \\ \left. \times \int_0^\infty \frac{\partial^2 \hat{\theta}_x}{\partial y_0^2} J_3(\sqrt{\kappa}(\lambda + \lambda)|y - y_0|) dy_0 \right] \quad \lambda \gg 1 \quad (43)$$

Also we neglect here higher order terms in  $(\lambda/\lambda)$  as well as the integral on the right-hand side. Then, it follows that

$$\lambda \hat{\theta}_x = \frac{1}{2\kappa\lambda} \frac{\partial^2 \hat{\theta}_x}{\partial y^2}$$

---

\* Appendix 1

(The solution of this equation  $A(\delta) \exp\{-\sqrt{2k\lambda\delta} y\}$  justifies above neglect of the integral.) Inverting this equation we obtain the Navier-Stokes equation for the present problem:

$$\frac{\partial g_x}{\partial t} = \frac{1}{2k\lambda} \frac{\partial^2 g_x}{\partial y^2} \quad (44)$$

Thus,  $1/2k\lambda$  is seen to correspond to the kinematic viscosity  $\nu$ .

As for the slip boundary condition, we set  $y=0$  in (21) and carry out integration by parts successively. Then we get

$$\left( \frac{1 + 2\delta/\lambda}{1 + \delta/\lambda} \hat{g}_x(0) - \frac{U}{\delta} \right) = \frac{1}{\sqrt{\pi k \lambda} (1 + \delta/\lambda)^{3/2}} \times \left[ \left( \frac{\partial \hat{g}_x}{\partial y} \right)_{y=0} + 2 \int_0^\infty \frac{\partial^2 \hat{g}_x}{\partial y^2} J_1(\sqrt{k}(\delta + \lambda)y_0) dy_0 \right] \quad (45)$$

If we assume here that Eq.(44) is still valid near the boundary, the second term on the right becomes higher order ( $O(\delta/\lambda)^{1/2}$ ) than the first. Therefore, neglecting higher order terms in ( $\delta/\lambda$ ), we have

$$\hat{g}_x(0) - \frac{U}{\delta} = \frac{1}{\sqrt{\pi k \lambda}} \left( \frac{\partial \hat{g}_x}{\partial y} \right)_{y=0}$$

Thus,

$$(g_x)_{y=0} - U = \frac{1}{\sqrt{\pi k \lambda}} \left( \frac{\partial g_x}{\partial y} \right)_{y=0} \quad (46)$$

This is a slip boundary condition with slip coefficient  $\delta = \frac{1}{\sqrt{\pi h} \lambda}$ . The above procedure is however only formal and does not provide a reasonable value for  $\delta$ . Further discussion will be made later of the slip coefficient.

We now return to the main problem to find the motion of the gas for  $\lambda t \gg 1$ , especially near the boundary. For this purpose we assume the solution in the form:

$$Q = Q_s + \frac{Z(\eta)}{\sqrt{\sigma}} + \dots \quad (47)$$

where  $Q_s$  corresponds to  $Q$  of the classical slip flow with  $\nu = 1/2h\lambda$  and  $\delta = (\beta / \sqrt{\pi h} \lambda)$ . That is,

$$Q_s = \left( \frac{1}{\sigma} - \frac{1}{\sqrt{\sigma}(\sqrt{\sigma} + \frac{1}{\beta} \sqrt{\frac{\pi}{2}})} \right) \exp\{-\sqrt{2\sigma} \eta\} \quad (48)$$

where  $\beta$  is a constant to be determined below. Substituting (47) into (29), we have

(right-hand side of (29))

$$\begin{aligned} &= \frac{1}{\sqrt{\pi}} \left\{ \frac{J_0((\sigma+1)\eta)}{\sigma} + \int_0^\infty Q_s J_{-1}((\sigma+1)|\eta-\eta_0|) d\eta_0 \right. \\ &\quad \left. + \frac{1}{\sqrt{\sigma}} \int_0^\infty Z J_{-1}((\sigma+1)|\eta-\eta_0|) d\eta_0 \right\} \end{aligned}$$

For small  $\sigma$  (and  $\sigma\eta$ ), each term becomes:

$$(\text{1st term}) = \frac{1}{\sigma} J_0(\eta) + O(1)$$

$$(\text{3rd term}) = \frac{1}{\sqrt{\sigma}} \left( \int_0^{\infty} Z \cdot J_1(|\eta - \eta_0|) d\eta_0 + O(\sigma) \right)$$

$$\begin{aligned} (\text{2nd term}) &= \left( \frac{1}{\sigma} - \frac{1}{\sqrt{\sigma} \left( \sqrt{\sigma} + \frac{1}{\beta} \sqrt{\frac{\pi}{2}} \right)} \right) \\ &\quad \times \int_0^{\infty} J_1((\sigma+1)|\eta - \eta_0|) \exp\{-\sqrt{2\sigma}\eta_0\} d\eta_0 \\ &= ( \quad ) \left\{ \int_0^{\eta} + \int_{\eta}^{\infty} \right\} \\ &= ( \quad ) \left[ \int_0^{\infty} \frac{\exp(-\xi^2)}{1 - \sqrt{2\sigma}\xi + \sigma} \left\{ \exp(-\sqrt{2\sigma}\eta) - \exp\left(-\frac{\sigma+1}{\xi}\eta\right) \right\} d\xi \right. \\ &\quad \left. + \int_0^{\infty} \frac{\exp(-\xi^2 - \sqrt{2\sigma}\eta)}{1 + \sqrt{2\sigma}\xi + \sigma} d\xi \right] \\ &= ( \quad ) \left[ \sqrt{\pi} \exp(-\sqrt{2\sigma}\eta) - J_0(\eta) - \sqrt{2\sigma} J_1(\eta) + O(\sigma) \right] \\ &= \sqrt{\pi} \left\{ \frac{1}{\sigma} - \frac{1}{\sqrt{\sigma} \left( \sqrt{\sigma} + \frac{1}{\beta} \sqrt{\frac{\pi}{2}} \right)} \right\} \exp(-\sqrt{2\sigma}\eta) \\ &\quad - \frac{1}{\sigma} J_0(\eta) + \frac{1}{\sqrt{\sigma}} \left[ \beta \sqrt{\frac{2}{\pi}} J_0(\eta) - \sqrt{2} J_1(\eta) \right] + O(1) \end{aligned}$$

From these results we find that the function  $Z(\eta)$  satisfies the following integral equation:

$$Z(\eta) = \sqrt{\frac{2}{\pi}} \left[ \frac{\beta}{\sqrt{\pi}} J_0(\eta) - J_1(\eta) \right] + \frac{1}{\sqrt{\pi}} \int_0^\infty Z(\eta_0) J_1(\eta - \eta_0) d\eta_0 \quad (49)$$

We shall denote by  $\beta_0$  the value of the parameter  $\beta$  for which the solution  $Z(\eta)$  of (49) tends to zero as  $\eta \rightarrow \infty$ . For arbitrary values of  $\beta$  and  $\eta$ ,

$$Z(\eta)_{\beta=\beta} - Z(\eta)_{\beta=\beta_0} = \sqrt{\frac{2}{\pi}} (\beta - \beta_0) \quad (50)$$

which is easily verified from (49). Thus

$$Z(\eta)_{\beta=\beta} \rightarrow \sqrt{\frac{2}{\pi}} (\beta - \beta_0) \quad \text{as } \eta \rightarrow \infty \quad (50a)$$

The field is therefore describable by the solution (48) for classical slip flow with  $\beta = \beta_0$  except in a layer adjacent to the plate with thickness of the order of mean free path. The correct value of the slip coefficient  $\delta$  may thus be taken as  $\delta = (\beta_0 / \sqrt{\pi k} \lambda)$ .

We now proceed to find  $Z(\eta)$  from Eq.(49). Our main concerns are the value  $\beta_0$  (for which  $Z(\eta) \rightarrow 0$  as  $\eta \rightarrow \infty$ ) and the behavior of  $Z(\eta)$  near the plate (for  $\eta \ll 1$ ).

In order to obtain the value  $\beta_0$ , we shall apply the moment method to Eq.(49). The solution  $Z(\eta)$  with  $\beta = \beta_0$  tends to zero as  $\eta \rightarrow \infty$  and may be considered to represent



molecular boundary effect. We shall assume  $Z(\eta)$  in the following form:

$$Z(\eta) = \alpha_0 J_0(\eta) + \alpha_1 J_1(\eta) + \alpha_2 J_2(\eta) + \alpha_3 J_3(\eta) \quad (51)$$

where  $\alpha_0$ ,  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$  are constants to be determined below. We substitute (51) into Eq.(49), multiply it by  $\eta^m$  ( $m=0, 1, 2, 3$  and  $4$ ) and integrate from

$\eta = 0$  to  $\eta = \infty$ . Then we have the following system of linear equations for  $\alpha_0$ ,  $\alpha_1$ ,  $\alpha_2$ ,  $\alpha_3$  and  $\beta_0$ :

$$\begin{aligned} & -J_{m+1}(0)\sqrt{\frac{2}{\pi}}\beta_0 + \sum_{i=0}^3 \left\{ \sqrt{\pi} J_{i+m+1}(0) \right. \\ & \left. - (E_{i,-1}^m + H_{m+i+1,-1}) \right\} \alpha_i = -\sqrt{2} J_{m+2}(0) \quad (52) \end{aligned}$$

where

$$\left. \begin{aligned} E_{n,-1}^m &= \iint_0^\infty \left( s^{n+1} \sum_{i=0}^m \gamma^{m-i} s^i \right) \exp\{-(s^2 + \gamma^2)\} ds d\gamma \\ H_{m+n,-1} &= \iint_0^\infty \frac{s^{m+n+1}}{s+\gamma} \exp\{-(s^2 + \gamma^2)\} ds d\gamma \end{aligned} \right\} \quad (53)$$

From these equations we obtain

$$\beta_0 = 1.801 \quad (54a)$$

$$\alpha_0 = 0.168, \quad \alpha_1 = 1.469, \quad \alpha_2 = -2.496, \quad \alpha_3 = 1.269. \quad (54b)$$

Corresponding value of  $\delta$  is

$$\delta = 0.90 / \ell \quad (55)$$

where  $\ell = (2/\sqrt{\pi} \hbar \lambda)$  is the mean free path. It is found that the numerical values in (54a) and (55) remain unchanged even if the series (51) is truncated by the 3rd term and only four moment equations are retained. The result by Gross et al.<sup>14</sup> for  $\delta$  is

$$\delta = 0.93 \ell$$

We next consider the behavior of  $Z(\eta)$  for  $\eta \ll 1$ .  $Z(\eta)$  may have some singularity at  $\eta = 0$ . The solution ((51) with (54b)) obtained above is an over-all approximation to  $Z(\eta)$  for  $0 < \eta < \infty$ , so that it is inadequate to accurate description of the local behavior of  $Z(\eta)$  for  $0 < \eta \ll 1$ . We must therefore return to Eq.(49).

It may then be expected that the integral in (49) is relatively insensible of the said singularity of  $Z(\eta)$  at  $\eta = 0$ . We may therefore substitute the over-all approximation (51) with (54b) for  $Z(\eta)$  there. Then, performing approximate evaluation of that integral for  $\eta \ll 1$  and expanding the first term also for small  $\eta$ , we arrive at the following result:

$$Z(\eta) \cong c_0 + c_1 \eta \log \eta + c_2 \eta + \dots \quad \eta \ll 1 \quad \left. \vphantom{Z(\eta)} \right\} (56)$$

$$c_0 = 0.436, \quad c_1 = 0.578, \quad c_2 = 0.396.$$

Now, taking inversion of (47), we find the gas velocity in the form:

$$\frac{g_x}{U} = \frac{g_{xs}}{U} + \frac{Z(\sqrt{k} \lambda y)}{\sqrt{\pi \lambda t}} + \dots \quad (57)$$

where  $g_{xs}$  is the velocity of the classical slip flow with  $v = (1/2k\lambda)$  and  $\delta = (\beta_0/\sqrt{\pi k} \lambda)$ . Namely

$$\begin{aligned} \frac{g_{xs}}{U} = & \operatorname{Erfc} \left\{ \frac{\sqrt{k} \lambda y}{\sqrt{2 \lambda t}} \right\} - \exp \left\{ \frac{\sqrt{\pi k} \lambda y}{\beta_0} + \frac{\pi \lambda t}{2 \beta_0^2} \right\} \\ & \times \operatorname{Erfc} \left\{ \frac{\sqrt{k} \lambda y}{\sqrt{2 \lambda t}} + \sqrt{\frac{\pi \lambda t}{2 \beta_0^2}} \right\} \\ & \left. \vphantom{\frac{g_{xs}}{U}} \right\} (58) \\ \operatorname{Erfc}[\xi] = & 1 - \frac{2}{\sqrt{\pi}} \int_0^\xi \exp\{-\zeta^2\} d\zeta \end{aligned}$$

For small values of  $\sqrt{k} \lambda y$  ( $= \frac{2}{\sqrt{\pi}} \frac{y}{\ell}$ ), we have from (56) the result

$$\frac{g_x}{U} \cong 1 + \frac{c_0 - \beta_0 \sqrt{\frac{2}{\pi}} + \sqrt{k} \lambda y \{c_1 \log(\sqrt{k} \lambda y) + c_2 - \sqrt{2}\}}{\sqrt{\pi \lambda t}} \quad (59)$$

and especially, the slip velocity on the plate:

$$\left(\frac{g_x}{U}\right)_{y=0} \cong 1 + \frac{c_0 - \beta_0 \sqrt{\frac{2}{\pi}}}{\sqrt{\pi \lambda t}} \quad (60)$$

The second term in (57) (and terms proportional to  $c_i$  in (59) or (60)) expresses a correction to the classical slip flow, which becomes significant near the boundary. It represents a kind of boundary layer with thickness of the order of mean free path, as noted by Gross et al.<sup>14</sup> In such a thin layer reflected molecules from the wall rarely have a chance of collision so that the effect of molecular boundary condition becomes important and the field has the same character as that for small time.

We now calculate the stress on the plate ( $y=0$ ). Substituting (47) into (31), we obtain

$$\begin{aligned} (P_{xy})_{y=0} = \frac{1}{\sqrt{\pi k} U} \left[ \frac{1}{2\sigma} - \int_0^\infty Q_\beta J_0((\sigma+1)\gamma_0) d\gamma_0 \right. \\ \left. - \frac{1}{\sqrt{\sigma}} \int_0^\infty Z J_0((\sigma+1)\gamma_0) d\gamma_0 \right] \quad (61) \end{aligned}$$

For small  $\sigma$ , each integral in the parentheses can be

easily calculated as follows:

$$\begin{aligned}
 \text{(2nd term)} &= \left( \frac{1}{\sigma} - \frac{1}{\sqrt{\sigma} \left( \sqrt{\sigma} + \frac{1}{\beta} \sqrt{\frac{\pi}{2}} \right)} \right) \\
 &\quad \times \int_0^{\infty} J_0((\sigma+1)\eta_0) \exp\{-\sqrt{2\sigma}\eta_0\} d\eta_0 \\
 &\cong \frac{1}{2} \left( \frac{1}{\sigma} - \frac{1}{\sqrt{\sigma} \left( \sqrt{\sigma} + \frac{1}{\beta} \sqrt{\frac{\pi}{2}} \right)} \right) - \frac{1}{2} \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{\sigma}}
 \end{aligned}$$

and

$$\begin{aligned}
 \text{(3rd term)} &\cong \frac{1}{\sqrt{\sigma}} \int_0^{\infty} Z(\eta_0) J_0(\eta_0) d\eta_0 \\
 &= \frac{1}{\sqrt{2\sigma}} \left( \frac{\beta}{\sqrt{\pi}} - \frac{\sqrt{\pi}}{2} \right)
 \end{aligned}$$

where the last integral is easily obtained by integrating (49) from 0 to  $\infty$ . Inserting these results in (61), we have

$$(P_{xy})_{y=0} \cong \frac{1}{\sqrt{2\pi h\sigma} U} \quad (62)$$

Inverting this, we get the stress on the plate in the form:

$$\left( \frac{P_{xy}}{\rho_0 U^2} \right)_{y=0} \cong \frac{1}{\sqrt{2\pi h\lambda t} U} \quad (\lambda t \gg 1) \quad (63)$$

This agrees with the classical results of Rayleigh and Schaaf with  $\nu = (1/2\lambda t)$  up to the order  $(t^{-1/2})^*$ . In this connection it may be noted that the result due to Gross et al.<sup>14</sup> involves a correction to the classical formula even in the order  $t^{-1/2}$ , giving about 10% smaller value than ours.

Fig.2 shows the distribution of the mean velocity of the gas for  $\lambda t = 0.2$ , while in Fig.3 and Fig.4 are respectively shown the slip velocity and the shear stress on the plate as functions of time. In the same figures, some results from the free molecular flow theory and the method of half-range moment by Gross et al.<sup>14</sup> are also shown by fine solid lines for comparison. Our solution contains a characteristic parameter  $\lambda$ . Therefore, to make the comparison, corresponding values of  $\lambda$  in other works must be calculated through a common parameter such as the mean free path or the kinematic viscosity. Relation between the mean free path and the kinematic viscosity differs with the models used (B-G-K, hard sphere, Maxwell molecule etc.) and the result of comparison depends on the common parameter used (mean free path or kinematic viscosity).  $\nu = 0.422 l/\sqrt{\pi}$  in the case of Gross et al.<sup>14</sup>, but  $\nu = \frac{\sqrt{\pi} l}{4\sqrt{\pi}}$  ( $\frac{\sqrt{\pi}}{4} = 0.443$ ) in our case. We have used, as the common parameter, the mean free path for  $\lambda t \ll 1$

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\* To this order the classical result is independent of slip coefficient  $\delta$ .

and the kinematic viscosity for  $\lambda t \gg 1$ . In free molecular flow,  $\lambda = 0$  and  $\lambda t$  is equal to zero for all  $t$ .

#### § 4. Effect of Temperature Variation of the Plate

Here is considered briefly the case when the velocity of the plate  $U(t)$  and its temperature  $T_w = T_0(1 + \tau_w(t))$  are arbitrary given functions of time  $t$ . Eqs.(10) to (13) are still valid in this case, so long as the assumptions for linearization are fulfilled. Multiplying Eq.(10) by  $F$  and integrating over all molecular velocities, we obtain the familiar continuity equation

$$\frac{\partial \omega}{\partial t} + \frac{\partial g_y}{\partial y} = 0 \quad (64)$$

which relates  $g_y$  with  $\omega$ .

Boundary condition at the wall is somewhat complicated: The molecules reflected at the wall have the Maxwellian distribution with velocity  $U$  and temperature  $T_w = T_0(1 + \tau_w)$ . Assuming  $U$  and  $\tau_w$  small and neglecting higher order terms we have

$$\phi_{y=0} = \phi_i(t) + 2k_u U(t) + \tau_w(t) \left( k v^2 - \frac{3}{2} \right) \quad (v > 0) \quad (65)$$

Here  $\phi_i(t)$  is a function of  $t$  alone and related to  $\phi_{y=0} \quad (v < 0)$  by the condition of no absorption at the wall. That is,

$$\int_{v>0} v \phi_{y=0} F dv + \int_{v<0} v \phi_{y=0} F dv = 0$$



and inserting (65) we have

$$\phi_i(t) = -2\sqrt{\pi\hbar} \int_{v<0} v \phi_{y=0} F dv - \frac{\tau_w}{2} \quad (66)$$

The condition at infinity is

$$\phi \rightarrow 0 \quad (v < 0) \quad \text{as } y \rightarrow \infty$$

We have also the initial condition:

$$\phi = 0 \quad \text{at } t = 0$$

From Eq.(10) with the initial and boundary conditions given above,  $\hat{\phi}$  can be expressed in terms of  $\hat{\omega}$ ,  $\hat{\mathcal{E}}$  and  $\hat{\tau}$  in a similar way as in § 2. The results are

$$\hat{\phi} = \frac{\lambda}{v} \int_{\infty}^y \left\{ \hat{\omega} + 2\hbar v \hat{\mathcal{E}} + \hat{\tau} \left( \hbar v^2 - \frac{3}{2} \right) \right\} \exp \left\{ \frac{\lambda + \lambda}{v} (y_0 - y) \right\} dy_0$$

( $v < 0$ ) (67a)

and

$$\hat{\phi} = \left[ \hat{\phi}_i + 2\hbar u \hat{U} + \hat{\tau}_w \left( \hbar v^2 - \frac{3}{2} \right) \right] \exp \left\{ -\frac{\lambda + \lambda}{v} y \right\}$$

$$+ \frac{\lambda}{v} \int_0^y \left\{ \hat{\omega} + 2\hbar v \hat{\mathcal{E}} + \hat{\tau} \left( \hbar v^2 - \frac{3}{2} \right) \right\} \exp \left\{ -\frac{\lambda + \lambda}{v} (y - y_0) \right\} dy_0$$

( $v > 0$ ) (67b)

where

$$\hat{\phi}_1 = -\frac{\hat{\tau}_w}{2} + 2\sqrt{\pi k} \lambda$$

$$\times \int_{v < 0}^{\infty} \left\{ \hat{\omega} + 2kv\hat{g}_y + \hat{\tau}(kv^2 - \frac{3}{2}) \right\} \exp\left\{\frac{4+2}{v}y_0\right\} dy_0 F dv \quad (68)^*$$

This (68) is obtained from (66) and (67a).

It will be seen from (67a, b) and (68) that

- 1)  $\hat{\phi}$  is a linear form of  $\hat{\omega}$ ,  $\hat{g}$ ,  $\hat{\tau}$ ,  $\hat{U}$  and  $\hat{\tau}_w$ .
- 2) Terms involving  $\hat{g}_x$  or  $\hat{U}$  in (67a, b) and (68) are odd functions of  $u$ .
- 3) Terms involving  $\hat{\omega}$ ,  $\hat{g}_y$ ,  $\hat{\tau}$  or  $\hat{\tau}_w$  are even functions of  $u$ .

Therefore if we take the first order moment

$$\hat{g}_x = \int u \hat{\phi} F dv$$

we obtain a functional relation involving  $\hat{g}_x$  and  $\hat{U}$  only. This is in fact an integral equation which reduces to Eq.(21) for constant  $U$ . On the other hand the two moments

$$\hat{\omega} = \int \hat{\phi} F dv$$

$$3(\hat{\tau} + \hat{\omega}) = 2k \int v^2 \hat{\phi} F dv$$

---

\* Note that the term  $[ (...) u \hat{g}_x ]$  has dropped out on integration over velocity space.

do not involve  $\hat{g}_x$  and  $\hat{U}$ , and these with (64) determine  $\hat{\omega}$ ,  $\hat{g}_y$  and  $\hat{\tau}$ . Further,  $\hat{g}_y$  may easily be eliminated and simultaneous integral equations for  $\hat{\omega}$  and  $\hat{\tau}$  are obtained. This problem will be discussed in Part 2.

## Part 2

Effect of Sudden Change of Wall Temperature  
in Rarefied Gas<sup>18</sup>

## Effect of Sudden Change of Wall Temperature in Rarefied Gas

### Summary

The response of a rarefied gas to abrupt change of temperature of bounding wall is discussed using the Bhatnager, Gross and Krook model of Boltzmann equation. The temperature change of the wall is assumed to be small and the governing equations as well as the boundary conditions are linearized. The density and temperature distributions in the gas are obtained for both short and long times. For short times the solution represents a perturbation to the linearized free molecular flow. At long times it involves essential differences from the corresponding solution based on the Navier-Stokes equation in a layer adjacent to the boundary with thickness of the order of mean free path. Numerical value of temperature jump distance is also obtained.

## § 1. Introduction

In Part 1 the writer discussed the behavior of a rarefied gas bounded by an infinite flat wall which is set impulsively into uniform motion in its own plane (Rayleigh problem), basing on the Bhatnager, Gross and Krook model<sup>1</sup> of the Boltzmann equation. Though the detailed analysis was carried out only for the case of constant wall temperature, it was noted there that the effect of possible variation of wall temperature would not be coupled with that of the motion of the wall so long as the assumptions for linearization are fulfilled (small velocity, small temperature variation, etc.). Therefore, the two effects may be analyzed separately and superimposed to form the answer to the generalized Rayleigh problem.

We thus deal in the present paper with the response of a rarefied gas to the abrupt small change of temperature of the bounding wall, basing on the B-G-K statistical model as in Part 1. The problem by itself is of considerable interest as one of the simplest and most fundamental problems of the rarefied gasdynamics, in which the boundary condition for temperature changes appreciably in a short time.

In § 2, the fundamental equations together with the boundary conditions are formulated for small variation of wall temperature. It turns out that no explicit form of

the velocity distribution function is needed to obtain relevant mean quantities. Thus, simultaneous integral equations for density and temperature of the gas are derived (Eqs.(17a,b)). Approximate solutions of these equations are obtained in § 3 for both short and long times. For a time which is short compared with mean collision period of gas molecules, the solution represents a perturbation to the linearized free molecular flow ((27) and (29)). It involves also at long times essential difference from the classical result based on the Navier-Stokes equation in a thin layer adjacent to the wall with thickness of the order of mean free path ((40) and (41)). The field outside this layer may be described by the classical solution. Numerical value of temperature jump distance is calculated ((38)).

## § 2. Fundamental Equation

There is a semi-infinite expanse of a dilute gas bounded by an infinite wall which is initially in equilibrium with the bounding gas at temperature  $T_0$ . We consider the behavior of the gas when the temperature of the wall is suddenly changed to  $T_w$  at time  $t=0$  and then is kept constant. We choose a Cartesian coordinate system  $(x,y,z)$  with its origin on the wall and  $y$ -axis normal to the wall. The gas is supposed to be in the region  $y>0$ . It may then be assumed that the field quantities are independent of  $x$  and  $z$  ( $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} = 0$ ).

In the present study we use, as in Part 1, the Boltzmann equation with the collisional model suggested by Krook et al.<sup>1</sup> Further, we assume that the temperature jump  $T_w - T_0$  of the wall given at  $t=0$  is so small that the fundamental equations and the boundary conditions may be linearized. Thus we obtain the following equations:

$$\frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial y} = \lambda \left\{ \omega - \phi + 2\hbar v g_y + \tau \left( \hbar v^2 - \frac{3}{2} \right) \right\} \quad (1)$$

$$\omega = \int \phi F dv \quad (2)^*$$

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\*) Integration  $\int (\dots) dv$  is hereafter carried out over the whole molecular velocity space unless otherwise stated.



$$g_y = \int v \phi F dv \quad (3)$$

$$\frac{3}{2h}(\omega + \tau) = \int v^2 \phi F dv \quad (4)$$

$$\left. \begin{aligned} F &= \left(\frac{h}{\pi}\right)^{\frac{3}{2}} \exp\{-h(u^2+v^2+w^2)\} \\ h &= \frac{m}{2kT_0} \end{aligned} \right\} \quad (5)$$

where  $m$  is the mass of a molecule,  $\mathbf{v}=(u,v,w)$  the velocity of a molecule,  $\rho_0 F(1+\phi)$  the mass density of molecules in 6 dimensional space  $(x,y,z;u,v,w)$ ,  $\rho_0(1+\omega)$  the mass density in physical space,  $\mathbf{g}=(0, g_y, 0)$  the gas velocity,  $T_0(1+\tau)$  the temperature,  $k$  the Boltzmann constant and  $\lambda$  a constant (collision frequency) related to mean free path  $\ell$  as  $\ell = 2/(\lambda\sqrt{\pi h})$ . Multiplying Eq.(1) by  $F$  and integrating over all molecular velocities, we obtain the relation between  $g_y$  and  $\omega$ , the hydrodynamic equation of continuity:

$$\frac{\partial \omega}{\partial t} + \frac{\partial g_y}{\partial y} = 0 \quad (6)$$

We also assume the diffuse reflection to describe the interaction of the gas with the bounding wall. Namely,

i) the reflected molecules have a Maxwellian distribution characterized by the velocity and temperature of the wall, and ii) the net mass flow at the wall is zero ( $\int v F(1+\phi) dv = 0$  at  $y=0$ ). Thus, the linearized boundary condition at the wall may be written as follows.

$$\phi_{y=0} = \phi_i(t) + \tau_w \left( \frac{1}{2} v^2 - \frac{3}{2} \right) \quad (v > 0) \quad (7)$$

where

$$\phi_i(t) = -2\sqrt{\pi k} \int_{v < 0} v \phi_{y=0} F dv - \frac{\tau_w}{2}, \quad \tau_w = \frac{T_w}{T_0} - 1$$

Further the condition at infinity is seen to be

$$\phi \rightarrow 0 \quad (v < 0) \quad \text{as } y \rightarrow \infty \quad (8)$$

while the initial condition of the problem is

$$\phi = 0 \quad \text{at } t = 0 \quad (9)$$

To solve the system of Eqs.(1) to (4) under the above conditions we apply the Laplace transformation ( $t \rightarrow s$ ,  $\phi \rightarrow \hat{\phi}$ , etc.) The transforms of (1) to (4), (6) and (7) may be written in the following forms:

$$s\hat{\phi} + v \frac{\partial \hat{\phi}}{\partial y} = \lambda \left\{ \hat{\omega} - \hat{\phi} + 2kv \hat{g}_y + \hat{\tau} \left( \frac{1}{2} v^2 - \frac{3}{2} \right) \right\} \quad (10)$$

$$\left. \begin{aligned} \hat{\phi}_{y=0} &= \hat{\phi}_i + \frac{\tau_w}{s} \left( \hbar v^2 - \frac{3}{2} \right) \quad (v > 0) \\ \hat{\phi}_i &= -2\sqrt{\pi\hbar} \int_{v < 0} v \hat{\phi}_{y=0} F dv - \frac{\tau_w}{2s} \end{aligned} \right\} (11)$$

$$\hat{\omega} = \int \hat{\phi} F dv \quad (12)$$

$$\hat{\phi}_y = \int v \hat{\phi} F dv \quad (13)$$

$$\frac{3}{2\hbar} (\hat{\tau} + \hat{\omega}) = \int v^2 \hat{\phi} F dv \quad (14)$$

$$s \hat{\omega} + \frac{\partial \hat{\phi}_y}{\partial y} = 0 \quad (15)$$

We first solve (10) for  $\hat{\phi}$  under the conditions (7) to (9) and express  $\hat{\phi}$  in terms of  $\hat{\omega}$ ,  $\hat{\phi}_y$  and  $\hat{\tau}$  in the forms

$$\left. \begin{aligned} \hat{\phi} &= \frac{\lambda}{v} \int_{\infty}^y \left\{ \hat{\omega} + 2\hbar v \hat{\phi}_y + \hat{\tau} \left( \hbar v^2 - \frac{3}{2} \right) \right\} \exp \left\{ \frac{s+\lambda}{v} (y_0 - y) \right\} dy_0 \\ &\quad (v < 0) \\ \hat{\phi} &= \left\{ \hat{\phi}_i + \frac{\tau_w}{s} \left( \hbar v^2 - \frac{3}{2} \right) \right\} \exp \left\{ -\frac{s+\lambda}{v} y \right\} \\ &\quad + \frac{\lambda}{v} \int_0^y \left\{ \hat{\omega} + 2\hbar v \hat{\phi}_y + \hat{\tau} \left( \hbar v^2 - \frac{3}{2} \right) \right\} \exp \left\{ \frac{s+\lambda}{v} (y_0 - y) \right\} dy_0 \\ &\quad (v > 0) \end{aligned} \right\} (16)$$

Then, eliminating  $\hat{\phi}$  and  $\hat{\rho}_y$  from (12), (14), (15) and (16), we arrive at the simultaneous integral equations for  $\hat{\omega}$  and  $\hat{\tau}$ .

$$\begin{aligned} \frac{\hat{\omega}}{\tau_w} = & \frac{1}{\sqrt{\pi}} \left\{ J_2(y) - J_0(y) \right\} \\ & + \lambda \left( \frac{\hbar}{\pi} \right)^{\frac{1}{2}} \left[ 2J_0(y) \int_0^{\infty} \left\{ \frac{\hat{\omega}}{\tau_w} (J_0(y_0) + \frac{2A}{A+\lambda} J_2(y_0)) + \frac{\hat{\tau}}{\tau_w} (J_2(y_0) - \frac{1}{2} J_0(y_0)) \right\} dy_0 \right. \\ & \left. + \int_0^{\infty} \left\{ \frac{\hat{\omega}}{\tau_w} (J_{-1}(y_1) + \frac{2A}{A+\lambda} J_1(y_1)) + \frac{\hat{\tau}}{\tau_w} (J_1(y_1) - \frac{1}{2} J_{-1}(y_1)) \right\} dy_1 \right] \end{aligned} \quad (17a)$$

$$\begin{aligned} \frac{\hat{\tau}}{\tau_w} = & \frac{1}{\sqrt{\pi}} \left\{ \frac{2}{3} J_4(y) - J_2(y) + J_0(y) \right\} \\ & + \lambda \left( \frac{\hbar}{\pi} \right)^{\frac{1}{2}} \left[ \frac{2}{3} \left\{ 2J_2(y) - J_0(y) \right\} \right. \\ & \times \int_0^{\infty} \left\{ \frac{\hat{\omega}}{\tau_w} (J_0(y_0) + \frac{2A}{A+\lambda} J_2(y_0)) + \frac{\hat{\tau}}{\tau_w} (J_2(y_0) - \frac{1}{2} J_0(y_0)) \right\} dy_0 \\ & + \frac{2}{3} \int_0^{\infty} \left\{ \frac{\hat{\omega}}{\tau_w} \left( \frac{2A}{A+\lambda} J_3(y_1) + \frac{\lambda}{A+\lambda} J_1(y_1) - \frac{1}{2} J_{-1}(y_1) \right) \right. \\ & \left. \left. \frac{\tau}{\tau_w} (J_3(y_1) - J_1(y_1) + \frac{5}{4} J_{-1}(y_1)) \right\} dy_1 \right] \end{aligned} \quad (17b)$$

where

$$Y = \sqrt{k}(\lambda + \lambda) y, \quad Y_0 = \sqrt{k}(\lambda + \lambda) y_0,$$

$$Y_1 = \sqrt{k}(\lambda + \lambda) |y - y_0|$$

and the functions  $J_n^*$  are defined by the integral

$$J_n(\xi) = \int_0^\infty \xi^n \exp\left\{-\xi^2 - \frac{\xi}{\xi}\right\} d\xi \quad (18)$$

In terms of the non-dimensional quantities

$$\eta = \sqrt{k} \lambda y, \quad (\eta_0 = \sqrt{k} \lambda y_0), \quad \sigma = \frac{\lambda}{\lambda} \quad (19)$$

$$\Omega = \lambda \hat{\omega}, \quad \mathcal{T} = \lambda \hat{\tau}$$

the resulting equations may be written in the forms:

$$\begin{aligned} \Omega(\eta, \sigma) = & \frac{1}{\sqrt{\pi}} \left[ \frac{\tau_w}{\sigma} \{J_2(Y) - J_0(Y)\} \right. \\ & + 2J_0(Y) \int_0^\infty \left\{ \Omega \cdot \left( J_0(Y_0) + \frac{2\sigma}{\sigma+1} J_2(Y_0) \right) + \mathcal{T} \cdot \left( J_2(Y_0) - \frac{1}{2} J_0(Y_0) \right) \right\} d\eta_0 \\ & \left. + \int_0^\infty \left\{ \Omega \cdot \left( J_1(Y_1) + \frac{2\sigma}{\sigma+1} J_1(Y_1) \right) + \mathcal{T} \cdot \left( J_1(Y_1) - \frac{1}{2} J_1(Y_1) \right) \right\} d\eta_0 \right] \end{aligned} \quad (20a)$$

---

\*) Appendix 1

$$\begin{aligned}
\mathcal{J}(\eta, \sigma) = & \frac{1}{\sqrt{\pi}} \left[ \frac{\tau_W}{\sigma} \left\{ \frac{2}{3} J_4(Y) - J_2(Y) + J_0(Y) \right\} \right. \\
& + \frac{2}{3} (2J_2(Y) - J_0(Y)) \\
& \times \int_0^\infty \left\{ \Omega \cdot \left( J_0(Y_0) + \frac{2\sigma}{\sigma+1} J_2(Y_0) \right) + \mathcal{J} \cdot \left( J_2(Y_0) - \frac{1}{2} J_0(Y_0) \right) \right\} d\eta_0 \\
& + \frac{2}{3} \int_0^\infty \left\{ \Omega \cdot \left( \frac{2\sigma}{\sigma+1} J_3(Y_1) + \frac{1}{\sigma+1} J_1(Y_1) - \frac{1}{2} J_{-1}(Y_1) \right) \right. \\
& \left. \left. + \mathcal{J} \cdot \left( J_3(Y_1) - J_1(Y_1) + \frac{5}{4} J_{-1}(Y_1) \right) \right\} d\eta_0 \right] \\
& (20 \text{ \#})
\end{aligned}$$

where

$$Y = (\sigma+1) \eta, \quad Y_0 = (\sigma+1) \eta_0$$

$$Y_1 = (\sigma+1) |\eta - \eta_0|$$

### § 3. Density and Temperature Distribution in the Gas\*

#### 3.1 Solution Suitable for Small Values of the Time

For short times (  $\lambda t \ll 1$  or  $\sigma \gg 1$  ), the state of affairs is expected to be close to those of the free molecular or collisionless flow (  $\lambda = 0$  ).

For free molecular flow, the right-hand sides of Eqs. (17a,b) degenerate into the respective first terms and we easily obtain the solution:

$$\omega = \frac{\tau_w}{2\sqrt{\pi}} \left[ \left( \frac{\sqrt{R}y}{t} \right) \exp \left\{ - \left( \frac{\sqrt{R}y}{t} \right)^2 \right\} - \int_{\frac{\sqrt{R}y}{t}}^{\infty} \exp \{ - \zeta^2 \} d\zeta \right] \quad (21a)$$

$$\tau = \frac{\tau_w}{\sqrt{\pi}} \left[ \frac{1}{3} \left( \frac{\sqrt{R}y}{t} \right)^3 \exp \left\{ - \left( \frac{\sqrt{R}y}{t} \right)^2 \right\} + \int_{\frac{\sqrt{R}y}{t}}^{\infty} \exp \{ - \zeta^2 \} d\zeta \right] \quad (21b)$$

In Fig. 1 are plotted  $\omega$  and  $\tau$  against  $(\sqrt{R}y/t)$ . The gas is seen to be rarefied on and near the wall, for outgoing molecules have higher velocity on the average than incoming molecules. For moderate and large  $y$ , the gas is condensed because reflected molecules with higher velocity overtake molecules with lower velocity which are reflected on the wall for  $t < 0$ .

Now, a refinement of the above results for short times

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\*) Once the density field is found, the velocity  $g_y$  is easily obtained from (6).

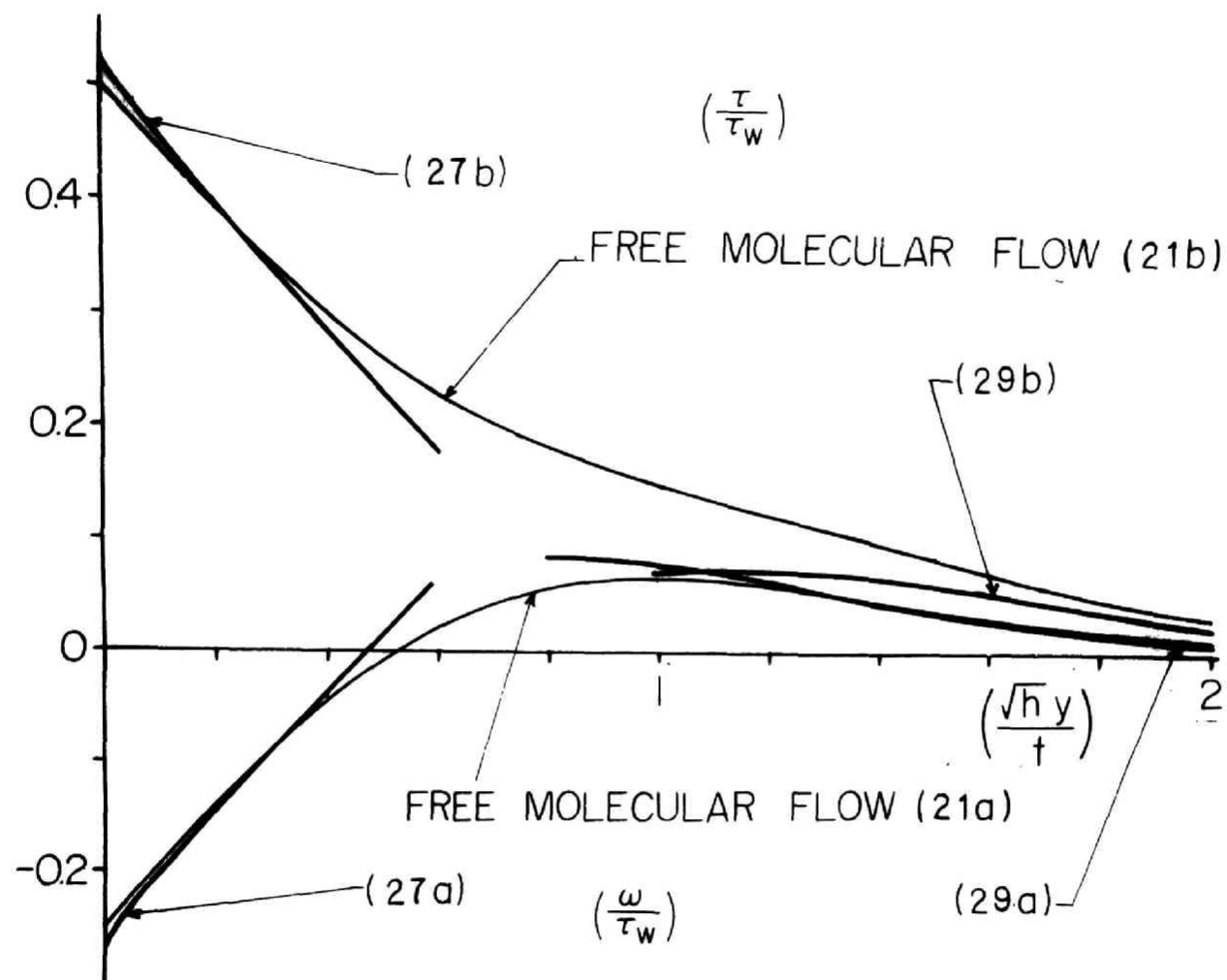


Fig. 1. The distributions of density and temperature in the gas ( $\lambda t = 0.2$ )



may be obtained by substituting the first terms (which represent the free molecular flow for  $\sigma \rightarrow \infty$ ) on the right-hand sides of Eqs.(20a,b) into  $\Omega$  and  $\mathcal{T}$  in the successive terms. Namely,

$$\begin{aligned} \Omega = \frac{\tau_w}{\sigma \sqrt{\pi}} & \left[ J_2(Y) - J_0(Y) \right. \\ & + \frac{2 J_0(Y)}{\sqrt{\pi}} \int_0^\infty \left\{ (J_2(Y_0) - J_0(Y_0)) \left( J_0(Y_0) + \frac{2\sigma}{\sigma+1} J_2(Y_0) \right) \right. \\ & \quad \left. + \left( \frac{2}{3} J_4(Y_0) - J_2(Y_0) + J_0(Y_0) \right) \left( J_2(Y_0) - \frac{1}{2} J_0(Y_0) \right) \right\} d\eta_0 \\ & + \frac{1}{\sqrt{\pi}} \int_0^\infty \left\{ (J_2(Y_0) - J_0(Y_0)) \left( J_1(Y_1) + \frac{2\sigma}{\sigma+1} J_1(Y_1) \right) \right. \\ & \quad \left. + \left( \frac{2}{3} J_4(Y_0) - J_2(Y_0) + J_0(Y_0) \right) \left( J_1(Y_1) - \frac{1}{2} J_1(Y_1) \right) \right\} d\eta_0 \left. \right] \end{aligned} \quad (22a)$$

$$\begin{aligned} \mathcal{T} = \frac{\tau_w}{\sigma \sqrt{\pi}} & \left[ \frac{2}{3} J_4(Y) - J_2(Y) + J_0(Y) + \frac{2}{3\sqrt{\pi}} \left\{ 2 J_2(Y) - J_0(Y) \right\} \right. \\ & \times \int_0^\infty \left\{ (J_2(Y_0) - J_0(Y_0)) \left( J_0(Y_0) + \frac{2\sigma}{\sigma+1} J_2(Y_0) \right) \right. \\ & \quad \left. + \left( \frac{2}{3} J_4(Y_0) - J_2(Y_0) + J_0(Y_0) \right) \left( J_2(Y_0) - \frac{1}{2} J_0(Y_0) \right) \right\} d\eta_0 \\ & + \frac{2}{3\sqrt{\pi}} \int_0^\infty \left\{ (J_2(Y_0) - J_0(Y_0)) \left( \frac{2\sigma}{\sigma+1} J_3(Y_1) + \frac{1}{\sigma+1} J_1(Y_1) - \frac{1}{2} J_1(Y_1) \right) \right. \\ & \quad \left. + \left( \frac{2}{3} J_4(Y_0) - J_2(Y_0) + J_0(Y_0) \right) \left( J_3(Y_1) - J_1(Y_1) + \frac{5}{4} J_1(Y_1) \right) \right\} d\eta_0 \left. \right] \end{aligned} \quad (22b)$$

Thus, the task is to evaluate the integral of the type

$$I = \int_0^{\infty} J_n(\alpha \eta_0) J_j(\alpha |\eta - \eta_0|) d\eta_0 \quad \alpha = (\sigma + 1) \quad (23)$$

a) Near field to the wall ( $\sqrt{k} y/t \ll 1$  or  $\sigma \eta \ll 1$ )

The integral (23) may be split into two parts:

$$I = \left( \int_0^{\eta} + \int_{\eta}^{\infty} \right) J_n(\alpha \eta_0) J_j(\alpha |\eta - \eta_0|) d\eta_0$$

In the first integral, the quantities  $\alpha \eta_0$  and  $\alpha(\eta - \eta_0)$  are both so small that we may use for  $J_n(\alpha \eta_0)$  and  $J_j(\alpha(\eta - \eta_0))$  respective series expansions around  $\eta = 0$  and  $\eta = \eta_0$ , and integrate term by term. Thus we have

$$\left. \begin{aligned} \int_0^{\eta} J_n(\alpha \eta_0) J_j(\alpha |\eta - \eta_0|) d\eta_0 &\cong -J_n(0) \left[ \eta \log \alpha \eta + \left( \frac{3}{2} \gamma - 1 \right) \eta \right] \\ \int_0^{\eta} J_n(\alpha \eta_0) J_j(\alpha |\eta - \eta_0|) d\eta_0 &\cong J_n(0) J_j(0) \eta \end{aligned} \right\} \quad n, j \geq 0 \quad (24)$$

where  $\gamma = 0.577\dots$  is Euler's constant. To evaluate the second integral, we insert for  $J_i$  its definition, change the order of integration and carry out some manipulation, obtaining

$$\int_{\eta}^{\infty} J_n(\alpha \eta_0) J_j(\alpha (\eta_0 - \eta)) d\eta_0 = \frac{1}{\alpha} (H_{n,j} - \alpha \eta H_{n-1,j} + \dots) \quad (25)$$

$$n+1, j \geq 0$$

where

$$\begin{aligned}
 H_{n,j} &= \int_0^\infty J_n(\eta_0) J_j(\eta_0) d\eta_0 = \iint_0^\infty \frac{\zeta^{n+1} \xi^{j+1}}{\zeta + \xi} \exp\{-\zeta^2 - \xi^2\} d\zeta d\xi \\
 &= \int_0^\infty r^{n+j+2} \exp(-r^2) dr \int_0^\infty \frac{\cos^{n+1} \theta \sin^{j+1} \theta}{\cos \theta + \sin \theta} d\theta \quad (26)
 \end{aligned}$$

Finally, taking the Laplace inversion of (22a,b) with the aid of (24), (25) and (19), we find the perturbation fields of density and temperature near the wall as follows:

$$\frac{\omega}{\tau_w} \cong -\frac{1}{4} + \frac{1}{\sqrt{\pi}} \left( \frac{\sqrt{R} y}{t} \right) + \lambda t \left\{ a_1 + a_2 \left( \frac{\sqrt{R} y}{t} \right) \log \left( \frac{\sqrt{R} y}{t} \right) + a_3 \left( \frac{\sqrt{R} y}{t} \right) \right\} \quad (27a)$$

$$\frac{\tau}{\tau_w} \cong \frac{1}{2} - \frac{1}{\sqrt{\pi}} \left( \frac{\sqrt{R} y}{t} \right) + \lambda t \left\{ \theta_1 + \theta_2 \left( \frac{\sqrt{R} y}{t} \right) \log \left( \frac{\sqrt{R} y}{t} \right) + \theta_3 \left( \frac{\sqrt{R} y}{t} \right) \right\} \quad (27b)$$

where

$$a_1 = -0.101, \quad a_2 = -0.307, \quad a_3 = -0.167$$

$$\theta_1 = 0.120, \quad \theta_2 = 0.290, \quad \theta_3 = 0.077$$

Plots of the results for  $\lambda t = 0.2$  are given also in Fig. 1. It will be seen that the gas is more rarefied and more heated than the free molecular flow case ( $\lambda = 0$ ) on and near the wall. This is because incoming molecules have lost some of mean velocity and obtained some heat due to collision with outgoing molecules (Fig.2).

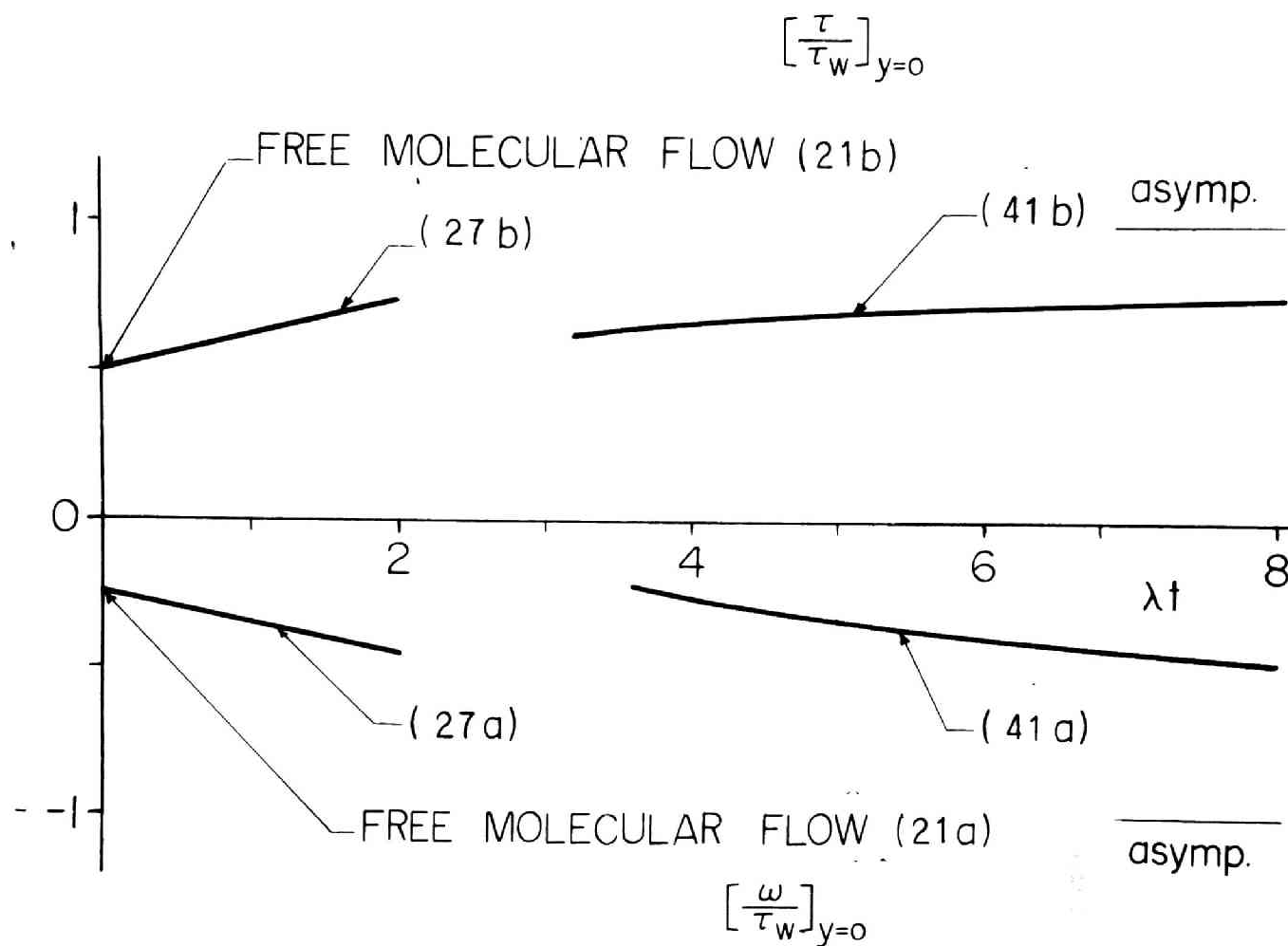


Fig. 2. Density and temperature of the gas on the wall  
 $( [\frac{\omega}{\tau_w}]_{y=0} \text{ and } [\frac{\tau}{\tau_w}]_{y=0} )$

b) Far field from the wall ( $\sqrt{R}y/t \gg 1$  or  $\sigma\eta \gg 1$ )

Breaking now the integral  $I$  into three parts:

$$I = \int_0^{\frac{1}{2}\eta} + \int_{\frac{1}{2}\eta}^{\eta} + \int_{\eta}^{\infty}$$

integrating by parts successively and taking into account the asymptotic behavior of  $J_i$  for  $\alpha\eta \gg 1$ , we get

$$I = \frac{1}{\alpha} \left[ 2 \left\{ J_{j+1}(0) J_n(\alpha\eta) + J_{j+3}(0) J_{n-2}(\alpha\eta) + \dots \right\} \right. \\ \left. + \left\{ J_{n+1}(0) J_j(\alpha\eta) + J_{n+2}(0) J_{j-1}(\alpha\eta) + \dots \right\} + \dots \right] \quad (28)$$

Then we take the Laplace inversions of (22a,b) with the aid of (19) and (28), obtaining the density and temperature fields as follows:

$$\frac{\omega}{\tau_w} \cong \frac{1}{2\sqrt{\pi}} \left[ 1 - \frac{1}{2} \left( \frac{t}{\sqrt{R}y} \right)^2 + \lambda t \left\{ -1 + \frac{13}{6} \left( \frac{t}{\sqrt{R}y} \right)^2 \right\} \right] \left( \frac{\sqrt{R}y}{t} \right) \exp \left\{ - \left( \frac{\sqrt{R}y}{t} \right)^2 \right\} \quad (29a)$$

$$\frac{\tau}{\tau_w} \cong \frac{1}{3\sqrt{\pi}} \left[ 1 + \lambda t \left\{ -1 + \left( \frac{t}{\sqrt{R}y} \right)^2 \right\} \right] \left( \frac{\sqrt{R}y}{t} \right)^3 \exp \left\{ - \left( \frac{\sqrt{R}y}{t} \right)^2 \right\} \quad (29b)$$

The results are plotted also in Fig. 1. The gas is less condensed and less heated for large  $y$  than the free molecular flow case, because outgoing molecules lose some of their (outgoing and thermal) velocities by collision with molecules going towards the wall before they reach the point in question.

It may be added here that the variation of wall temperature induces as is shown in Ref. 19 a pressure pulse (sound wave) which travels into the gas. The study of formation of this wave from the kinetic theory analysis will be of considerable interest. The phenomena should be contained in our Eqs.(17a,b) or (20a,b). It is however difficult to obtain complete solution for whole space and time. We have here to content ourselves with the information of very initial stage of the process.

### 3.2 Solution for Large Values of the Time

We now proceed to the problem to find the behavior of the gas for  $\lambda t \gg 1$ , especially near the boundary. For this purpose we assume the solution in the forms:

$$\Omega(\eta, \sigma) = \Omega_s + \frac{\Omega_1(\eta)}{\sqrt{\sigma}} + \dots \quad (30 a)$$

$$\mathcal{T}(\eta, \sigma) = \mathcal{T}_s + \frac{\mathcal{T}_1(\eta)}{\sqrt{\sigma}} + \dots \quad (30 b)$$

where  $\Omega_s$  and  $\mathcal{T}_s$  correspond to  $\Omega$  and  $\mathcal{T}$  of the classical results based on the Navier-Stokes equation with kinematic viscosity  $\nu = 1/2k\lambda$ , thermal conductivity  $k = 5k_B\nu/2m$  (corresponding to B-G-K model), specific heat at constant pressure  $c_p = 5k/2m$ , and temperature jump distance  $\delta_T = \sqrt{\pi} c \ell/2$ . That is,

$$\begin{aligned}\Omega_S &= A_1 e^{R_1 \eta} + A_2 e^{R_2 \eta} \\ \mathcal{I}_S &= r_1 A_1 e^{R_1 \eta} + r_2 A_2 e^{R_2 \eta}\end{aligned}\tag{31}$$

where

$$\begin{aligned}R_1 &= -\sqrt{2\sigma}(1+O(\sigma)), \quad R_2 = -\sqrt{\frac{6}{5}}\sigma(1+O(\sigma)), \\ A_1 &= -\frac{\tau_w}{\sigma} \left[ 1 - \left( \frac{2}{\sqrt{5}} + c\sqrt{2} \right) \sqrt{\sigma} + O(\sigma) \right], \\ A_2 &= \sqrt{\frac{3}{5}} \frac{\tau_w}{\sqrt{\sigma}} \left[ 1 + O(\sigma) \right], \\ r_1 &= -(1+O(\sigma)), \quad r_2 = \frac{2}{3}(1+O(\sigma))\end{aligned}$$

and  $c$  is a constant to be determined below. On substituting (30 a,b) in (20a,b) and carrying out some reduction, we find for  $\Omega_1$  and  $\mathcal{I}_1$  the following integral equations:

$$\begin{aligned}\Omega_1(\eta) &= \frac{\tau_w}{\sqrt{\pi}} \left\{ -\sqrt{2} \left( J_3(\eta) - \frac{3}{2} J_1(\eta) \right) + \sqrt{2} c \left( J_2(\eta) - J_0(\eta) \right) \right\} \\ &\quad + \frac{2}{\sqrt{\pi}} J_0(\eta) \int_0^\infty \left\{ \Omega_1 \cdot J_0(\eta_0) + \mathcal{I}_1 \cdot \left( J_2(\eta_0) - \frac{1}{2} J_0(\eta_0) \right) \right\} d\eta_0 \\ &\quad + \frac{1}{\sqrt{\pi}} \int_0^\infty \left\{ \Omega_1 \cdot J_1(|\eta - \eta_0|) + \mathcal{I}_1 \cdot \left( J_1(|\eta - \eta_0|) - \frac{1}{2} J_{-1}(|\eta - \eta_0|) \right) \right\} d\eta_0.\end{aligned}\tag{32 a)}$$

$$\begin{aligned}
\mathcal{T}_1(\eta) = & \frac{\tau_w}{\sqrt{\pi}} \left[ -\frac{2\sqrt{2}}{3} \left\{ J_5(\eta) - 2J_3(\eta) + \frac{7}{4}J_1(\eta) \right\} + \sqrt{2}c \left\{ \frac{2}{3}J_4(\eta) - J_2(\eta) + J_0(\eta) \right\} \right] \\
& + \frac{2}{3\sqrt{\pi}} \left\{ 2J_1(\eta) - J_0(\eta) \right\} \int_0^\infty \left\{ \Omega_1 \cdot J_0(\eta_0) + \mathcal{T}_1 \cdot \left( J_2(\eta_0) - \frac{1}{2}J_0(\eta_0) \right) \right\} d\eta_0 \\
& + \frac{2}{3\sqrt{\pi}} \int_0^\infty \left\{ \Omega_1 \cdot \left( J_1(|\eta - \eta_0|) - \frac{1}{2}J_{-1}(|\eta - \eta_0|) \right) \right. \\
& \quad \left. + \mathcal{T}_1 \cdot \left( J_3(|\eta - \eta_0|) - J_1(|\eta - \eta_0|) + \frac{5}{4}J_{-1}(|\eta - \eta_0|) \right) \right\} d\eta_0
\end{aligned}
\tag{32b}$$

We shall denote by  $c_0$  the value of the parameter  $c$  for which  $\mathcal{T}_1$  tends to zero as  $\eta \left( = \frac{2y}{\ell\sqrt{\pi}} \right) \rightarrow \infty$ . For arbitrary values of  $c$  and  $\eta$

$$\begin{aligned}
[\Omega_1]_{c=c} - [\Omega_1]_{c=c_0} &= 0 \\
[\mathcal{T}_1]_{c=c} - [\mathcal{T}_1]_{c=c_0} &= \sqrt{2}(c - c_0)\tau_w
\end{aligned}
\tag{33}$$

which are easily verified from (32). Thus

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\*) If  $\Omega_1$  and  $\mathcal{T}_1$  are solutions of Eqs.(32a,b), then  $\Omega_1 + (\text{arbitrary constant})$  and  $\mathcal{T}_1$  may also be solutions of (32a,b). However, the only one which tends to zero as  $\eta \rightarrow \infty$  satisfies the boundary condition (8) (c.f. (16), (30) and (31)).



$$[\mathcal{T}_1]_{c=c} \rightarrow \sqrt{2}(c-c_0)\tau_w \quad (\text{as } \eta \rightarrow \infty) \quad (33a)$$

The fields are therefore describable by the solution (31) with  $c=c_0$  except in a layer adjacent to the wall with thickness of the order of mean free path. The correct value of the temperature jump distance may thus be taken as  $\delta_T = (\sqrt{\pi}c_0 l/2)$ .

We now proceed to find  $\Omega_1$  and  $\mathcal{T}_1$  from Eqs.(32a,b). Our main concerns are the value  $c_0$  for which  $\mathcal{T}_1 \rightarrow 0$  as  $\eta \rightarrow \infty$  and the behavior of  $\Omega_1$  and  $\mathcal{T}_1$  near the wall ( $\eta \ll 1$ ). In order to obtain the value  $c_0$ , we shall apply the moment method to Eqs.(32a,b). The solutions  $\Omega_1$  and  $\mathcal{T}_1$  with  $c=c_0$  tends to zero as  $\eta \rightarrow \infty$  and may be considered to represent molecular boundary effect. We thus assume  $\Omega_1$  and  $\mathcal{T}_1$  in the following forms:

$$\frac{\Omega_1}{\tau_w} = \alpha_0 J_0 + \alpha_1 J_1 + \alpha_2 J_2 + \alpha_3 J_3 \quad (34a)$$

$$\frac{\mathcal{T}_1}{\tau_w} = \beta_0 J_0 + \beta_1 J_1 + \beta_2 J_2 + \beta_3 J_3 \quad (34b)$$

where  $\alpha_i$ 's and  $\beta_i$ 's are constants to be determined below. We substitute (34a,b) into Eqs.(32a,b), multiply them by  $\eta^m$  ( $m=0, 1, 2, 3$  and  $4$ ) and integrate from  $\eta=0$  to  $\infty$ . Then we have simultaneous linear equations for  $\alpha_i$ ,  $\beta_i$ , and  $c_0$ :

$$\begin{aligned}
& \{J_{m+3}(0) - J_{m+1}(0)\} \sqrt{\frac{2}{\pi}} c_0 \\
& + \sum_{i=0}^3 \left[ -J_{i+m+1}(0) + \frac{2}{\sqrt{\pi}} J_{m+1}(0) H_{i,0} + \frac{1}{\sqrt{\pi}} (E_{i,-1}^m + H_{i+m+1,-1}) \right] \alpha_i \\
& + \sum_{i=0}^3 \left[ \frac{2}{\sqrt{\pi}} J_{m+1}(0) \left( H_{i,2} - \frac{1}{2} H_{i,0} \right) \right. \\
& \quad \left. + \frac{1}{\sqrt{\pi}} \left( E_{i,1}^m - \frac{1}{2} E_{i,-1}^m + H_{i+m+1,1} - \frac{1}{2} H_{i+m+1,-1} \right) \right] \beta_i \\
& = \sqrt{\frac{2}{\pi}} \left\{ J_{m+4}(0) - \frac{3}{2} J_{m+2}(0) \right\} \quad (35a)^*
\end{aligned}$$

$$\begin{aligned}
& \left\{ \frac{2}{3} J_{m+5}(0) - J_{m+3}(0) + J_{m+1}(0) \right\} \sqrt{\frac{2}{\pi}} c_0 \\
& + \frac{2}{3\sqrt{\pi}} \sum_{i=0}^3 \left[ \left\{ 2J_{m+3}(0) - J_{m+1}(0) \right\} H_{i,0} + \left( E_{i,1}^m - \frac{1}{2} E_{i,-1}^m + H_{i+m+1,1} - \frac{1}{2} H_{i+m+1,-1} \right) \right] \alpha_i \\
& + \sum_{i=0}^3 \left[ -J_{i+m+1}(0) + \frac{2}{3\sqrt{\pi}} \left\{ 2J_{m+3}(0) - J_{m+1}(0) \right\} \left( H_{i,2} - \frac{1}{2} H_{i,0} \right) \right. \\
& \quad \left. + \left( E_{i,3}^m - E_{i,1}^m + \frac{5}{4} E_{i,-1}^m + H_{i+m+1,3} - H_{i+m+1,1} + \frac{5}{4} H_{i+m+1,-1} \right) \right] \beta_i \\
& = \frac{2}{3} \sqrt{\frac{2}{\pi}} \left\{ J_{m+6}(0) - 2J_{m+4}(0) + \frac{7}{4} J_{m+2}(0) \right\} \quad (35b)
\end{aligned}$$

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\*) The equation for  $m=0$  vanishes identically.

where

$$E_{n,i}^m = \iint_0^\infty \xi^{n+1} \zeta^{i+1} \left( \sum_{i=0}^m \xi^{m-i} \zeta^i \right) \exp \{-\xi^2 - \zeta^2\} d\xi d\zeta$$

$$= \sum_{i=0}^m J_{m+n+1-i}(0) \cdot J_{j+i+1}(0) = E_{j,n}^m \quad (35)$$

$$H_{i,j} = \iint_0^\infty \frac{\xi^{i+1} \zeta^{j+1}}{\xi + \zeta} \exp \{-\xi^2 - \zeta^2\} d\xi d\zeta \quad (26)$$

Solving these equations we obtain

$$c_0 = 1.302 \quad (37a)^*$$

$$\left. \begin{aligned} \alpha_0 &= 2.996, & \alpha_1 &= -19.534, & \alpha_2 &= 27.288, \\ \alpha_3 &= -10.449 \\ \beta_0 &= -4.193, & \beta_1 &= 25.485, & \beta_2 &= -34.466, \\ \beta_3 &= 13.029 \end{aligned} \right\} \quad (37b)$$

Corresponding value of  $\delta\tau$  is

$$\delta\tau = 1.154 l \quad (38)$$

where  $l = 2/\lambda\sqrt{\pi h}$  is the mean free path.

We next consider the behaviors of  $\Omega_1$  and  $\mathcal{T}_1$  for  $\eta \ll 1$ .

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\*) It is found that the numerical values in (37a) and (38) converge rapidly. That is, we obtain  $c_0 = 1.300$  (or 1.304) and  $\delta\tau = 1.152 l$  (or 1.156  $l$ ) if the series (34a,b) are truncated by the 2nd (or 3rd) term and only five (or seven) moment equations ( $m=0, 1$  and 2 (or 0, 1, 2 and 3)) are retained.

$\Omega_1$  and  $\mathcal{T}_1$  may have some singularity at  $\eta = 0$ . Their expressions (34a,b) with (37b) obtained above are approximations for over-all domain  $0 < \eta < \infty$ , so that they are inadequate to accurate description of the local behaviors of  $\Omega_1$  and  $\mathcal{T}_1$  for  $0 < \eta \ll 1$ . We must therefore return to Eqs.(32a,b). Since the integrals in (32a,b) are relatively insensible of the said singularities, we may substitute (34a,b) for  $\Omega_1$  and  $\mathcal{T}_1$  there. Thus, performing approximate evaluation of the integrals for  $\eta \ll 1$ , and expanding other terms also for small  $\eta$ , we arrive at the following results:

$$\begin{aligned}\frac{\Omega_1}{\tau_w} &\cong d_0 + d_1 \eta \log \eta + d_2 \eta + \dots \\ \frac{\mathcal{T}_1}{\tau_w} &\cong e_0 + e_1 \eta \log \eta + e_2 \eta + \dots\end{aligned}\tag{39}^*$$

where

$$\begin{aligned}d_0 &= -0.479, & d_1 &= -0.666, & d_2 &= -0.575 \\ e_0 &= 0.623, & e_1 &= 0.680, & e_2 &= 0.359\end{aligned}$$

Finally, taking the Laplace inversions of (30a,b) we find the density and temperature fields in the forms:

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\*) Since  $d_1$ ,  $d_2$ ,  $e_1$  and  $e_2$  are sensitive to the  $\Omega_1(0)$  and  $\mathcal{T}_1(0)$ , their values are corrected by making use of  $d_0$  and  $e_0$  which converge rapidly.

$$\omega = \omega_s + \frac{\Omega_1(\eta)}{\sqrt{\pi\lambda t}} + \dots \quad (40a)$$

$$\tau = \tau_s + \frac{\mathcal{T}_1(\eta)}{\sqrt{\pi\lambda t}} + \dots \quad (40b)$$

where  $\omega_s$  and  $\tau_s$  are classical solutions. Inserting (39) in (40a,b) we have for small values of  $\eta = 2y/l\sqrt{\pi}$  the results

$$\frac{\omega}{\tau_w} \cong -1 + \frac{\left(\frac{\sqrt{15}}{3} + c_0\sqrt{2} + d_0\right) + d_1\eta \log \eta + (\sqrt{2} + d_2)\eta}{\sqrt{\pi\lambda t}} \quad (41a)$$

$$\frac{\tau}{\tau_w} \cong 1 + \frac{(-c_0\sqrt{2} + e_0) + e_1\eta \log \eta + (-\sqrt{2} + e_2)\eta}{\sqrt{\pi\lambda t}} \quad (41b)$$

The second terms in (40a,b) (and terms proportional to  $d_i$  and  $e_i$  in (41a,b)) express corrections to the classical results which become significant near the boundary. They represent a kind of boundary layer with thickness of the order of mean free path. In such a thin layer reflected molecules from the wall rarely have a chance of collision so that the effect of molecular boundary condition becomes important and the field has the same character as that for short times.

In Fig.2 are shown the variations of the density and temperature of the gas on the wall as functions of time.

## Appendix

# Appendix 1 Functions $J_n(x)$ <sup>16,17</sup>

Here are listed some important properties of functions  $J_n(x)$  which are frequently used in the present study.

1) Definition

$$J_n(x) = \int_0^\infty \xi^n \exp\left\{-\xi^2 - \frac{x}{\xi}\right\} d\xi \quad (x \geq 0)$$

2)

$$\frac{dJ_n(x)}{dx} = -J_{n-1}(x)$$

3)

$$J_{2n}(0) = \int_0^\infty \xi^{2n} \exp\{-\xi^2\} d\xi = \frac{1 \cdot 3 \cdots (2n-1)}{2^{n+1}} \sqrt{\pi}$$

$$J_{2n+1}(0) = \int_0^\infty \xi^{2n+1} \exp\{-\xi^2\} d\xi = \frac{n!}{2}$$

$$J_0(0) = \frac{\sqrt{\pi}}{2}$$

where  $n$  is a positive integer.

4) Series expansion

$$J_0(x) = \frac{\sqrt{\pi}}{2}$$

$$- \sum_{i=0}^{\infty} \frac{(-1)^i \left\{ \frac{1}{2} \psi(i+1) + \psi(2i+2) - \log x \right\}}{i! (2i+1)!} x^{2i+1}$$

$$- \sqrt{\pi} \sum_{i=0}^{\infty} \frac{(-2)^i x^{2i+2}}{1 \cdot 3 \cdot 5 \cdots (2i+1) [(2i+2)!]}$$

$$\psi(i+1) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{i} - \gamma$$

$$\psi(1) = -\gamma$$

where  $\gamma = 0.577\dots$  is Euler's constant. The expansion of  $J_n(x)$  where  $n$  is an arbitrary integer is easily obtained from 4) by the aid of 2) and 3).

5) Asymptotic expansion

$$J_n(x) \cong \sqrt{\frac{\pi}{3}} \left(\frac{x}{2}\right)^{\frac{n}{3}} \exp\left[-3\left(\frac{x}{2}\right)^{\frac{2}{3}}\right] \quad (x \rightarrow \infty)$$



## Appendix 2 B-G-K model<sup>3,15</sup>

Here is briefly discussed the relation between the standard Boltzmann equation and the simplified equation suggested by Krook et al. The Boltzmann equation for a monoatomic gas with no external forces can generally be written in the form

$$\frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{x}} = m g(f) - f \mathcal{A}(f) \quad (1)$$

where  $m$  is the mass of a molecule,  $f/m$  the number density of molecules in physical ( $\mathbf{x}$ ) and velocity ( $\mathbf{v}$ ) space, and  $g(f)$  and  $(f/m)\mathcal{A}(f)$  are well-known non-linear integral terms giving, respectively, the number of molecules gained and lost as a result of molecular collision per unit volume and unit time at  $(\mathbf{x}, t; \mathbf{v})$ . The main difficulty in handling the Boltzmann equation arises from the complicated nature of the collision terms  $g(f)$  and  $(f/m)\mathcal{A}(f)$ . One way of simplifying these terms is discussed in the following.

We start with the loss term  $(f/m)\mathcal{A}(f)$ . Consider the collision of two molecules with velocity  $\mathbf{v}$  (molecule (0)) and  $\mathbf{v}_1$  (molecule (1)). We assume here that the force of molecular interaction is of short range and proportional to  $r^{-5}$  where  $r$  is the distance between the molecules (Maxwell molecule). The molecule (0) interacts with the molecules (1) for any large approach distance.

Thus, the loss term diverges if it is treated separately from the gain term and all interactions are taken into account. Most of them are, however, too weak to cause appreciable change in  $v$  or momentum. When we consider the loss term separately, as in the following, we have to use a cut off (e.g. in momentum or energy) properly.\* We consider here a cut off in momentum change. For collisions with large approach distances the change of momentum  $\delta M$  of a molecule may be calculated from impulse consideration.

The result shows:  $\delta M \sim \frac{\text{const.}}{b^4 g} \quad g = |v - v_1|$

where  $b$  is the approach distance. On the other hand, the characteristic change of momentum  $\Delta M$  of a molecule is  $mg$  when two molecules with relative speed  $g$  collide. We now disregard such weak interactions that the ratio  $\delta M / \Delta M \sim \text{const.} / b^4 g^2$  is smaller than a certain (small) value. Thus, the cut off value  $b_0$  of the approach distance is  $b_0 \sim \text{const.} / g^{\frac{1}{2}}$ . Then the effective collision cross-section  $\sigma_0 = \pi b_0^2$  is equal to

$$\sigma_0 = \frac{m}{\chi g}$$

where  $\chi$  is a constant which depends on the sort of molecule. We therefore obtain the rate of collision of a

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\*) It is also noted that  $dv$  is small but finite to assure the treatment by distribution function.

molecule (0) with any other molecule (considering a cylinder of height  $g$  and base area  $\sigma_0 = m/xg$ ) as

$$\iiint_{-\infty}^{\infty} \frac{m}{xg} \times g \times \frac{f(x, t, v_i)}{m} dv_i = \frac{\rho}{x}$$

where  $\rho$  is the density of the gas. Thus,  $\frac{\rho f}{x m} dx dv$  is the number of molecules (0) which take part in collision per unit time in the volume element  $dx dv$  under consideration and leave there. Namely,

$$\left(\frac{f}{m}\right) \mathcal{A}(f) = \frac{\rho f}{x m}$$

The same relationship is obtained also from consideration of change of energy.

We next consider the gain term  $g(f)$ . We must survey the distribution of molecules after collision. This is more difficult to handle. We here content ourselves with a rough approximation. Important requirements for collision term  $g(f) - \left(\frac{f}{m}\right) \mathcal{A}(f)$  are i) it vanishes when  $f$  is a locally Maxwellian distribution, ii) mass, momentum and kinetic energy in  $dx$  are conserved in collision of molecules (short range force). Now putting  $g(f) = \frac{\rho}{x m} F(x, t, v)$ , we see from ii) that  $F$  is a distribution having the same density, velocity and temperature as  $f$ . The functional form of  $F$  generally depends on  $f$ . The distribution of molecules which have experienced collision is, however, fairly close to equilibrium compared with their initial

distribution, as shown by Kogan.\* We may therefore consider, as a first approximation, that  $F$  is independent of  $f$ , that is,  $F=f_0$  where  $f_0$  is the Maxwellian distribution corresponding to  $f$ . Thus we obtain as the gain term

$$g(f) = \frac{\rho f_0}{\chi m}$$

To sum up, the Boltzmann equation (1) may be written in the following simple approximate form (B-G-K equation)

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = \frac{\rho}{\chi} (f_0 - f)$$

where  $\chi$  is a parameter which in general may depend on the state of the gas. This equation appears very simple but contains some moments of  $f$  in  $f_0$  which are not known beforehand and is still a nonlinear integro-differential equation.

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<sup>15</sup>  
\*) Kogan calculated the distributions of molecules after collision for some distributions very far from equilibrium before collision in the case of a rigid sphere molecule.

### Appendix 3 Kinetic Theory Analysis of Diffusion of Discontinuity in Flow Velocity<sup>4</sup>

Problems in which state variables (e.g. density, velocity, etc.) change considerably in a short range (comparable to mean free path) in a gas are of main interest in rarefied gasdynamics as well as those in which the conditions of a solid boundary in a gas change in a short time (comparable to mean collision period of gas molecules). As a simple and fundamental example of the former phenomena, we here try to investigate how a plane of initial discontinuity in gas velocity diffuses on the basis of the Boltzmann equation with B-G-K model.

At time  $t=0^*$  the gas is assumed to have uniform velocity  $U$  in  $x$ -direction in the region  $y>0$  and  $-U$  in  $y<0$ , uniform density  $\rho_0$  and uniform temperature  $T_0$ , and to be in equilibrium in respective regions. The plane at  $y=0$  of initial discontinuity in gas velocity is released to diffuse for  $t>0$ . We further assume that the initial velocity  $U$  is much less than the velocity of sound in the gas so that the fundamental equations as well as the initial and boundary conditions may be linearized. Then, we can show that the density and temperature of the gas remain constant for  $t>0$ . More generally, we can treat the velocity in  $x$ -direction independently of the density, temperature and ve-

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\*)  $t$  is time and  $(x,y,z)$  Cartesian coordinate system.

locity in y-direction in the problem where the initial density and temperature are also not uniform\*. Accordingly, our result given below may represent the field of velocity in x-direction in this more general problem.

On the basis of the B-G-K model, the linearized kinetic equations become

$$\left. \begin{aligned} \frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial y} &= \lambda (-\phi + 2h u g_x) \\ g_x &= \iiint_{-\infty}^{\infty} u \phi F dv \\ F &= \left(\frac{h}{\pi}\right)^{\frac{3}{2}} \exp\{-h(u^2 + v^2 + w^2)\} \quad h = \frac{m}{2kT_0} \end{aligned} \right\} (1)$$

where  $(\rho_0/m) F(1+\phi)$  is the velocity distribution function,  $g_x$  the x-component of the gas velocity,  $m$  the mass of a molecule,  $\mathbf{v}=(u,v,w)$  the molecular velocity,  $k$  the Boltzmann constant, and  $\lambda$  a constant (collision frequency) related to the mean free path  $\ell$  as follows:  $\ell = 2/\lambda\sqrt{\pi h}$ . The initial condition is

$$\begin{aligned} \phi &= 2h u U & (y > 0) \\ &= -2h u U & (y < 0) \end{aligned} \quad \text{at } t=0 \quad (2)$$

while the boundary conditions are

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\*) The proof may be given in quite a similar way as in Part 1. Nonuniformity must be small to assure the linearization.

$$\begin{aligned}
 \phi &= 2k_u U \quad (v < 0) \quad \text{as } y \rightarrow \infty \\
 &= -2k_u U \quad (v > 0) \quad \text{as } y \rightarrow -\infty
 \end{aligned}
 \tag{3}$$

$\phi$  is continuous at  $y=0$ .

From (1), (2) and (3) we obtain, after some reduction, the following integral equation for  $\bar{g}_x$  :

$$\begin{aligned}
 \frac{\bar{g}_x}{U} &= \frac{\pm 1}{s+\lambda} \left\{ 1 - \frac{2}{\sqrt{\pi}} J_0(\sqrt{k}(s+\lambda)|y|) \right\} \\
 &\quad + \lambda \sqrt{\frac{k}{\pi}} \int_{-\infty}^{\infty} \frac{\bar{g}_x}{U} J_1(\sqrt{k}(s+\lambda)|y-y_0|) dy_0
 \end{aligned}
 \tag{4}$$

where the bar over a letter indicates the Laplace transform ( $t \rightarrow s$ ), the upper sign holds for  $y > 0$  and the lower for  $y < 0$ , and the functions  $J_n'$  are defined by the integral

$$J_n(\xi) = \int_0^{\infty} \zeta^n \exp\left[-\zeta^2 - \frac{\xi}{\zeta}\right] d\zeta$$

For short times ( $\lambda t \ll 1$ ), the state of affairs is expected to be close to the free molecular flow ( $\lambda = 0$ ). For free molecular flow, the right-hand side of Eq.(4) degenerates into its first term and we easily obtain the solution

$$\frac{\bar{g}_x}{U} = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{k}y/t} \exp(-\zeta^2) d\zeta
 \tag{5}$$

The solution for short times for our problem may be obtained by adding a perturbation to this result. Thus, we obtain

$$\frac{g_x}{U} \cong \frac{2\sqrt{h}y}{\sqrt{\pi}t} \left[ 1 + \lambda t \left\{ \sqrt{2} \log(\sqrt{2}+1) - 1 \right\} \right] \quad \left| \frac{\sqrt{h}y}{t} \right| \ll 1 \quad (6a)$$

$$\begin{aligned} \frac{g_x}{U} \cong 1 - \left[ 1 - \frac{1}{2} \left( \frac{t}{\sqrt{h}y} \right)^2 - \lambda t \left\{ 1 - \frac{3}{2} \left( \frac{t}{\sqrt{h}y} \right)^2 \right\} \right] \\ \times \frac{t}{\sqrt{\pi h} y} \exp \left\{ - \left( \frac{\sqrt{h}y}{t} \right)^2 \right\} \quad \left| \frac{\sqrt{h}y}{t} \right| \gg 1 \quad (6b) \end{aligned}$$

The effect of molecular collision at the initial stage appears in the following way. Molecules coming from the region  $y > 0$  have lost some of their average velocity in x-direction by collision with molecules which have emerged from  $y < 0$ , while molecules from  $y < 0$  have obtained the average velocity by collision with molecules from  $y > 0$ . For  $y > 0$  the latter effect dominates the former. Thus, the flow is less decelerated than the free molecular flow there. In other words, the diffusion (or mixing) is slowed down by molecular collisions. It is also noted that the result (6a) does not contain any nonanalytic term such as  $(\sqrt{h}y/t) \log(\sqrt{h}y/t)$  in contrast to the case in which a solid boundary exists as in Rayleigh flow. Fig.1 shows the distribution of gas velocity against  $\sqrt{h}y/t$  for  $\lambda t = 0.4$  (the result for free molecular flow is also shown for comparison).



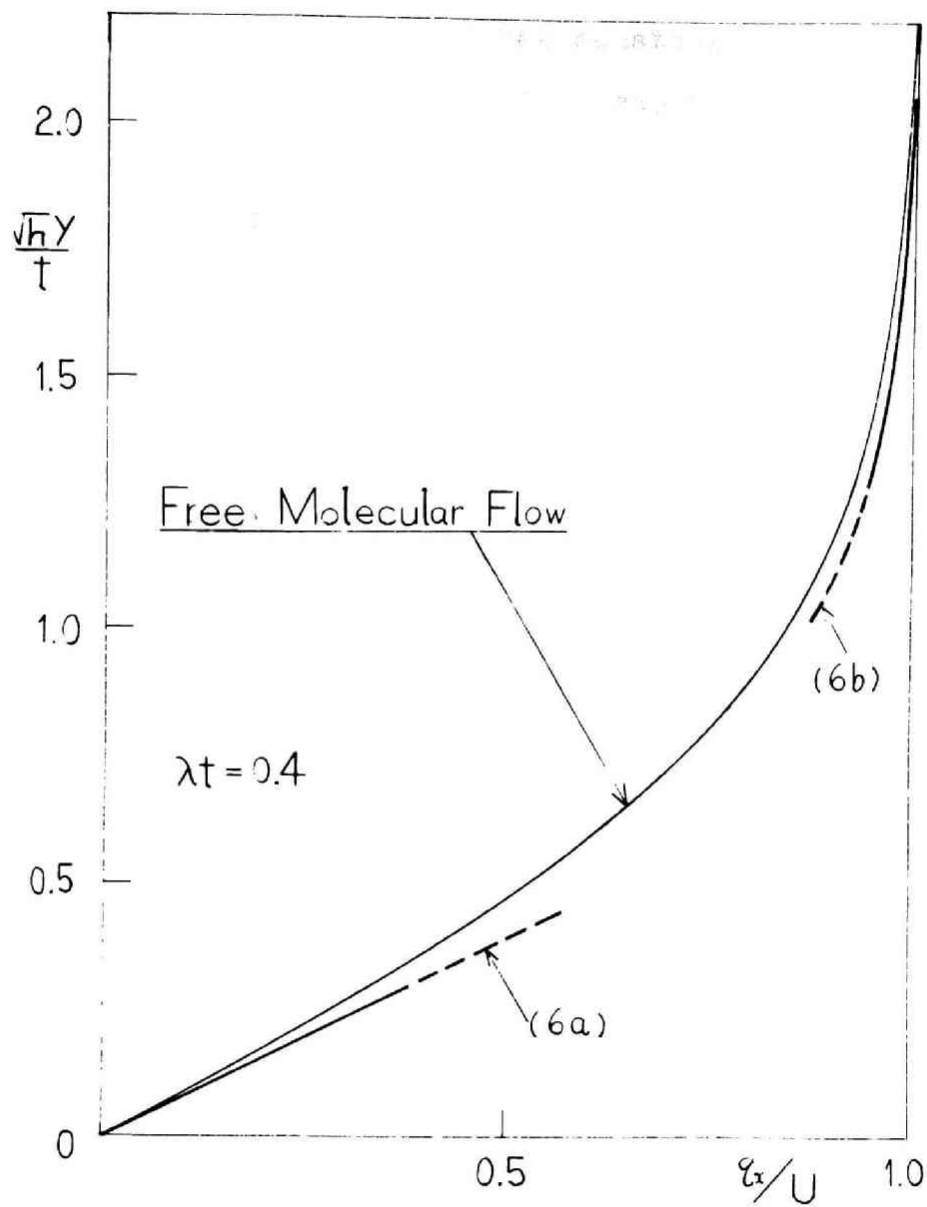


Fig. 1. The distribution of the velocity in the gas  
( $\lambda t = 0.4$ )

For  $\lambda t \gg 1$ , the field may be described by the classical result based on the Navier-Stokes equation. That is, the classical solution with kinematic viscosity  $\nu = 1/2k\lambda$  (corresponding to B-G-K model)

$$\frac{\theta x}{U} = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{k\lambda y}/\sqrt{2\lambda t}} \exp(-\xi^2) d\xi \quad (7)$$

may be seen to satisfy the integral equation (4) by direct substitution if we assume  $t^{-1}$  and  $\sqrt{k\lambda y}/t$  small\* and neglect these and higher order terms. (Note that  $k\lambda y^2/t$  may not necessarily be small.) The detailed analysis is similar as in Part 1. The above result may be compared with the result in Rayleigh flow where a solid boundary plays fundamental role and a correction proportional to  $t^{-\frac{1}{2}}$  to the classical result appears representing a kind of boundary layer with thickness of the order of mean free path. Roughly speaking, for  $\lambda t \gg 1$ , in the region of viscous diffusion ( $y \lesssim (2t/k\lambda)^{\frac{1}{2}}$ ) which is much smaller than that of (free) molecular diffusion ( $y \lesssim t k^{-\frac{1}{2}}$ ),

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\*) Further study is required to clear the detailed behavior for  $y \gtrsim k^{-\frac{1}{2}} t$ . The region  $y \sim k^{-\frac{1}{2}} t$  is interesting especially in the case where pressure pulse (sound wave) travels into the gas as in Part 2 since the ridge of the sound pulse is advancing there.

mixing and collision of molecules occur adequately to assure the validity of the Navier-Stokes equation except near a solid boundary which does not exist in the present problem. It may also be seen from (6) and (7)\* that the effective speed of diffusion is slowed down from  $O(k^{-\frac{1}{2}})$  to  $O\{(2k\lambda t)^{-\frac{1}{2}}\}$  as time progresses.

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\*) Note that the solution (6) depends on  $\sqrt{k} y / t$  except for a small perturbation and that (7) is a function of  $\{\sqrt{k} \lambda y / (2\lambda t)^{\frac{1}{2}}\}$ .

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